MAT8021, Algebraic Topology

Assignment 4

Due in-class on Tuesday, March 16

1. (The Five Lemma) Suppose

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \delta\downarrow & & \epsilon\downarrow \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5 \end{array}$$

   where the rows are exact and the squares commute. Suppose α, β, δ, ϵ are isomorphisms. Show that γ is an isomorphism.

2. Prove a stronger version of the Five Lemma: If β and δ in the above diagram are injective, and α is surjective, show that γ is injective. Give the dual statement (whose proof is of course essentially the same).

3. (Formal Mayer–Vietoris sequence) Suppose that there is a map of long exact sequence as follows:

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & F_{n+1} & \longrightarrow & A_n & \longrightarrow & B_n & \longrightarrow & F_n & \longrightarrow & A_{n-1} & \longrightarrow & \cdots \\ & & \sim\downarrow & & \downarrow & & \downarrow & & \sim\downarrow & & \downarrow & & \\ \cdots & \longrightarrow & G_{n+1} & \longrightarrow & C_n & \longrightarrow & D_n & \longrightarrow & G_n & \longrightarrow & C_{n-1} & \longrightarrow & \cdots \end{array}$$

   Here all the maps $F_n \to G_n$ are isomorphisms. Show that there is a long exact sequence

$$\cdots \to D_{n+1} \to A_n \to B_n \oplus C_n \to D_n \to A_{n-1} \to \cdots$$

   (Define the maps first.)

4. Suppose X is a CW complex with finitely many cells. Define the Euler characteristic $\chi(X)$ to be the number of even-dimensional cells minus the number of odd-dimensional cells.

If F is *any* field, show that

$$\chi(X) = \sum_i (-1)^i \dim_F \big(H_i(X;F)\big)$$

(Hint: Express this in terms of the ranks of the boundary maps.)