MAT8021, Algebraic Topology

Assignment 12

Due in-class on Tuesday, May 18

1. Suppose X is a path-connected (based) space, M is a compact orientable manifold, and $f : S^1 \wedge X \to M$ is a map inducing an isomorphism on homology with integer coefficients. Show that X has the same homology as a sphere S^n. Hint: Besides Poincaré duality, need to use the fact that cup product is 0 on $S^1 \wedge X$ (recall relative cup products).

2. Suppose M is a compact oriented $4n$-dimensional manifold with $H^{2n}(M; \mathbb{Z})$ torsion free. Poincaré duality gives us a pairing

$$x, y \mapsto x \cdot y \colon H^{2n}(M; \mathbb{Z}) \times H^{2n}(M; \mathbb{Z}) \to \mathbb{Z}$$

which is distributive and satisfies $x \cdot y = y \cdot x$. If $e_1, \ldots, e_g$ are a basis of $H^{2n}(M; \mathbb{Z})$, there is a symmetric matrix $A = (a_{ij})$ such that $e_i \cdot e_j = a_{ij}$.

If we choose a different basis $f_k = \sum_i c_{ki} e_i$, we get a different matrix B. Express B in terms of A using matrix multiplication.

3. Suppose M and N are n-dimensional compact manifolds with orientations $[M] \in H_n(M; \mathbb{Z})$ and $[N] \in H_n(N; \mathbb{Z})$. We define the *degree* of a map $f \colon M \to N$ to be the unique integer a such that $f_*([M]) = a[N]$.

Show that the degree of a map $\mathbb{CP}^2 \to \mathbb{CP}^2$ is always a square.

4. One statement of Poincaré duality for manifolds with boundary says: If M is a compact manifold with boundary ∂M, there are isomorphisms

$$D \colon H^p(M; \mathbb{Z}/2) \to H_{n-p}(M, \partial M; \mathbb{Z}/2)$$

(M is not necessarily orientable).

Use this to show that there is no compact 3-dimensional manifold W with boundary $\partial W = \mathbb{RP}^2$.