MAT8021, Algebraic Topology

Assignment 10

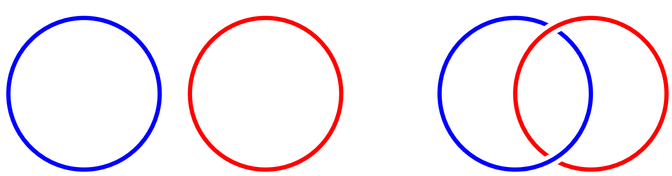
Due in-class on Tuesday, April 27

1. In class, we stated the fact that the homology and cohomology, as graded abelian groups, of a knot complement are independent of the knot (and hence cannot distinguish different knots, whereas the cup product does). Calculate that
$$H^*(\mathbb{R}^3 \setminus K) \cong \begin{cases} \mathbb{Z} & * = 0,1,2 \\ 0 & \text{otherwise} \end{cases}$$
and
$$H^*(S^3 \setminus K) \cong \begin{cases} \mathbb{Z} & * = 0,1 \\ 0 & \text{otherwise} \end{cases}$$
by taking $K = S^1$ to be the unknot. (Hint: You can think of S^3 as obtained from the solid 3-ball D^3 by identifying the upper hemisphere with the lower hemisphere, analogous to the 2-dimensional case. From this, show that $S^3 \setminus S^1$ is homeomorphic to a solid torus.)

2. The left-hand portion of the picture below is a union L_1 of two disconnected circles in $\mathbb{R}^3$. Show that the complement $X = \mathbb{R}^3 \setminus L_1$ retracts down onto $S^2 \vee S^2 \vee S^1 \vee S^1$. Use this to show that the cup product of any two elements in $H^1(X)$ is zero.

3. The right-hand portion of the picture is a link L_2 in $\mathbb{R}^3$. Show that the complement $Y = \mathbb{R}^3 \setminus L_2$ has the same cohomology as the space X from the previous problem. (Possible hint: Show that the space retracts down onto something gotten by gluing two tori together along $S^1 \vee S^1$. Don't appeal to any major duality theorems like Alexander duality.)

4. Show that the cup product of the two generators in $H^1(Y)$ is nonzero. (Possible hint: Compare it with a torus.)



Two links in $\mathbb{R}^3$