MAT8021, Algebraic Topology

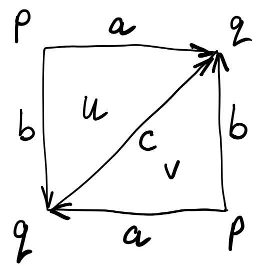
Final Exam

Friday, June 11, 2021

4:30-6:30 pm

There are 6 questions in total.

1. (a) (5 points) Construct a Δ-set whose geometric realization is the 2-dimensional real projective space $\mathbb{RP}^2$. (In particular, write down its lists X_n of n-simplices for $0 \le n \le 2$, together with all face maps $\partial_n^i : X_n \to X_{n-1}$ for $1 \le n \le 2$ and $0 \le i \le n$. A picture will be helpful.)



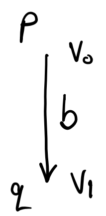
$X_0 = \{p, q\}$

$X_1 = \{a, b, c\}$

$X_2 = \{u, v\}$

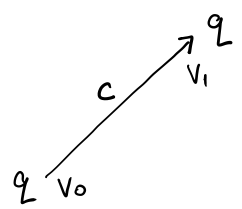
$\partial_1^0(a) = q$

$\partial_1^1(a) = p$

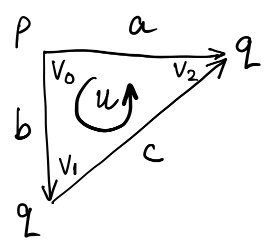


$\partial_1^0(b) = q$

$\partial_1^1(b) = p$



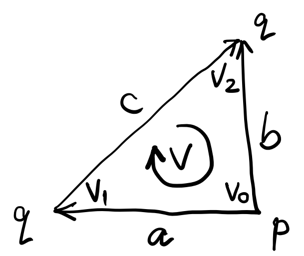
$\partial_1^0(c) = \partial_1^1(c) = q$



$\partial_2^0(u) = c$

$\partial_2^1(u) = a$

$\partial_2^2(u) = b$



$\partial_2^0(v) = c$

$\partial_2^1(v) = b$

$\partial_2^2(v) = a$

(b) (5 points) Compute the (simplicial) homology groups $H_n(\mathbb{RP}^2;\mathbb{Z})$ using the Δ-complex structure from part (a).

From part (a) we obtain

$$C_0 = \mathbb{Z}\cdot p \oplus \mathbb{Z}\cdot q$$

$$\partial_0(p) = \partial_0(q) = 0$$

$$C_1 = \mathbb{Z}\cdot a \oplus \mathbb{Z}\cdot b \oplus \mathbb{Z}\cdot c$$

$$\partial_1(a) = q - p$$

$$\partial_1(b) = q - p$$

$$\partial_1(c) = q - q = 0$$

$$C_2 = \mathbb{Z}\cdot u \oplus \mathbb{Z}\cdot v$$

$$\partial_2(u) = c - a + b$$

$$\partial_2(v) = c - b + a$$

Therefore

$$Z_0 = \mathbb{Z}\cdot p \oplus \mathbb{Z}\cdot q$$

$$B_0 = \mathbb{Z}\cdot(q-p)$$

$$Z_1 = \mathbb{Z}\cdot(a-b) \oplus \mathbb{Z}\cdot c$$

$$B_1 = \mathbb{Z}\cdot(c-a+b) \oplus \mathbb{Z}\cdot(c-b+a)$$

$$Z_2 = B_2 = 0$$

and so

$$H_0 = Z_0/B_0 \cong (\mathbb{Z}\cdot p \oplus \mathbb{Z}\cdot q)/(p=q) \cong \mathbb{Z}$$

$$H_1 = Z_1/B_1 \cong \frac{\mathbb{Z}\cdot d \oplus \mathbb{Z}\cdot c}{\mathbb{Z}\cdot(c-d)\oplus\mathbb{Z}\cdot(c+d)}, \quad \text{where } d := a-b$$

$$\cong \frac{\mathbb{Z}\cdot d}{\mathbb{Z}\cdot 2d} \cong \mathbb{Z}/2$$

$$H_n = 0, \quad n \geqslant 2$$

(c) (10 points) Use this Δ-complex structure to compute $H^*(\mathbb{RP}^2; \mathbb{Z}/2)$ together with the cup product on it.

From part (b) we obtain

$$C^0 = \mathbb{Z}/2 \cdot p^* \oplus \mathbb{Z}/2 \cdot q^*$$

$$C^1 = \mathbb{Z}/2 \cdot a^* \oplus \mathbb{Z}/2 \cdot b^* \oplus \mathbb{Z}/2 \cdot c^*$$

$$C^2 = \mathbb{Z}/2 \cdot u^* \oplus \mathbb{Z}/2 \cdot v^*$$

with

$$\partial_1 : C_1 \longrightarrow C_0$$

$$\begin{bmatrix} \\ \\ \end{bmatrix} \longmapsto \begin{array}{c} \\ p \\ q \end{array} \!\!\! \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \end{array} \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

transpose $\Longrightarrow$ mod 2

$$\delta^0 : C^0 \longrightarrow C^1$$

$$\begin{bmatrix} \\ \\ \\ \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \\ \\ \end{bmatrix}$$

$$\partial_2 : C_2 \longrightarrow C_1$$

$$\begin{bmatrix} \\ \\ \\ \end{bmatrix} \longmapsto \begin{array}{c} \\ a \\ b \\ c \end{array} \!\!\! \begin{array}{c} \begin{array}{cc} u & v \end{array} \\ \begin{bmatrix} -1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \end{array} \begin{bmatrix} \\ \\ \end{bmatrix}$$

transpose $\Longrightarrow$ mod 2

$$\delta^1 : C^1 \longrightarrow C^2$$

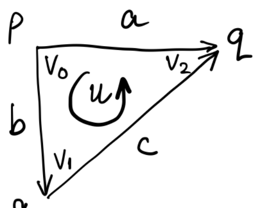
$$\begin{bmatrix} \\ \\ \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

Therefore

$$
\begin{aligned}
Z^0 &= \mathbb{Z}/2 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
B^0 &= 0 \\
Z^1 &= \mathbb{Z}/2 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \oplus \mathbb{Z}/2 \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \\
B^1 &= \mathbb{Z}/2 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\
Z^2 &= \mathbb{Z}/2 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \oplus \mathbb{Z}/2 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
B^2 &= \mathbb{Z}/2 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\end{aligned}
$$

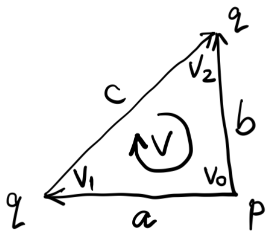
and so

$$
\begin{aligned}
H^0 &\cong \mathbb{Z}/2 \\
H^1 &\cong \mathbb{Z}/2 \cdot [\alpha] = \mathbb{Z}/2, \text{ where } \alpha = b^* + c^* \\
H^2 &\cong \mathbb{Z}/2 \\
H^n &= 0, \quad n > 2
\end{aligned}
$$



Moreover, for the graded multiplicative structure,

$$(\alpha \cup \alpha)(u) = \alpha(b) \cdot \alpha(c) = 1 \cdot 1 = 1$$



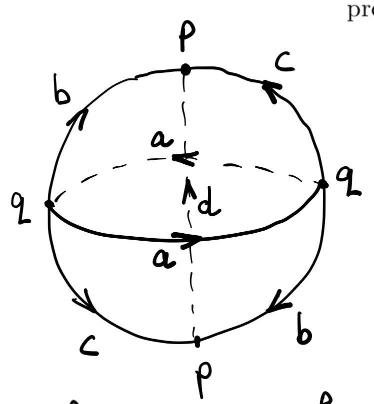
$$(\alpha \cup \alpha)(v) = \alpha(a) \cdot \alpha(c) = 0 \cdot 1 = 0$$

$$\Rightarrow \quad \alpha \cup \alpha = u^* \notin B^2$$

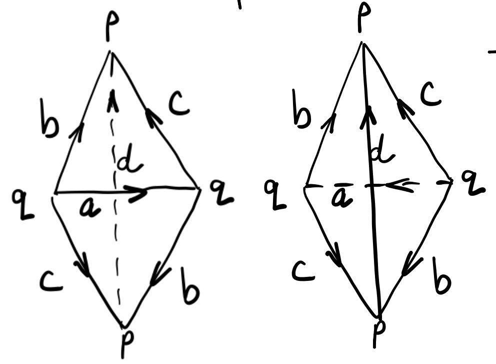
$$\Rightarrow \quad [\alpha] \cup [\alpha] = [u^*]$$

$$\Rightarrow \quad H^*(\mathbb{R}P^2; \mathbb{Z}/2) \cong \mathbb{Z}/2[x]/(x^3), \quad \text{where } x = [\alpha] \in H^1.$$

2. (a) (10 points) Recall that we may view $\mathbb{RP}^3$ as the lens space $L(2,1)$ which is formed by gluing together two tetrahedra. Use this Δ-complex structure to compute $H^*(\mathbb{RP}^3;\mathbb{Z}/2)$ together with the cup product on it.



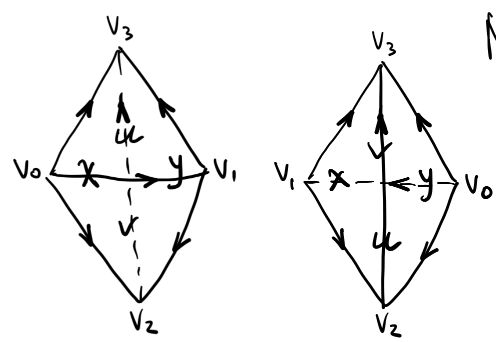
We can obtain $\mathbb{RP}^3$ by gluing antipodal points on the boundary S^2 of D^3, or equivalently, by gluing two tetrahedra as in the picture.



front s   back t

This gives

$$X_0 = \{p, q\}$$
$$X_1 = \{a, b, c, d\}$$
$$X_2 = \{u, v, x, y\}$$
$$X_3 = \{s, t\}$$



Moreover, we have in s

$$\partial a = 0 \qquad \partial u = c - b + a$$
$$\partial b = \partial c = p - q \qquad \partial v = b - c + a \qquad \partial s = y - x + u - v$$
$$\partial x = d - b + c$$
$$\partial d = 0 \qquad \partial y = d - c + b$$

and further in t, $\partial t = x - y + v - u$

Dually and modulo 2, we obtain

$$\delta p^* = \delta q^* = b^* + c^*$$
$$\delta a^* = u^* + v^*$$
$$\delta b^* = \delta c^* = u^* + v^* + x^* + y^*$$
$$\delta d^* = x^* + y^*$$
$$\delta u^* = \delta v^* = \delta x^* = \delta y^* = s^* + t^*$$

Therefore

$$Z^0 = \mathbb{Z}/2 \cdot (p^* + q^*)$$
$$B^0 = 0$$
$$Z^1 = \mathbb{Z}/2 \cdot (a^* + b^* + d^*) \oplus \mathbb{Z}/2 \cdot (b^* + c^*)$$
$$B^1 = \mathbb{Z}/2 \cdot (b^* + c^*)$$
$$Z^2 = \mathbb{Z}/2 \cdot (u^* + v^*) \oplus \mathbb{Z}/2 \cdot (x^* + y^*) \oplus \mathbb{Z}/2 \cdot (u^* + x^*)$$
$$B^2 = \mathbb{Z}/2 \cdot (u^* + v^*) \oplus \mathbb{Z}/2 \cdot (x^* + y^*)$$
$$Z^3 = \mathbb{Z}/2 \cdot s^* \oplus \mathbb{Z}/2 \cdot t^*$$
$$B^3 = \mathbb{Z}/2 \cdot (s^* + t^*)$$

and so

$$H^0 = \mathbb{Z}/2 \cdot [p^* + q^*]$$
$$H^1 = \mathbb{Z}/2 \cdot [a^* + b^* + d^*] =: \mathbb{Z}/2 \cdot [\alpha]$$
$$H^2 = \mathbb{Z}/2 \cdot [u^* + x^*] =: \mathbb{Z}/2 \cdot [\beta]$$
$$H^3 = \mathbb{Z}/2 \cdot [s^*]$$

Moreover, since

$$\alpha \cup \alpha\, (\beta^*) = \alpha(a + c) \cdot \alpha(c + d) = (1+0) \cdot (0+1) = 1$$
$$\alpha \cup \alpha \cup \alpha\, (s) = \alpha(a) \cdot \alpha(b) \cdot \alpha(d) = 1 \cdot 1 \cdot 1 = 1$$

we get $H^*(\mathbb{RP}^3; \mathbb{Z}/2) \cong \mathbb{Z}/2\,[w]/(w^4)$, where $w = [\alpha] \in H^1$.

(b) (5 points) Let m be a positive integer. Give a presentation for $H^*(\mathbb{RP}^m; \mathbb{Z}/2)$ as a graded ring in terms of generators and relations. No justification is needed.

$H^*(\mathbb{RP}^m; \mathbb{Z}/2) \cong \mathbb{Z}/2\,[w]/(w^{m+1})$ with $|w| = 1$.

3. (a) (10 points) Describe a CW-complex structure on the real Grassmannian Gr(2, 4).

Gr(2,4) has points 2-dimensional subspaces of $\mathbb{R}^4$. They are in bijection with 2×4 matrices with linearly independent row vectors.

Turn these into reduced echelon forms:

$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ gives a 0-cell

$\begin{bmatrix} 0 & 1 & a & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ gives a 1-cell

$\begin{bmatrix} 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \end{bmatrix}$ $\begin{bmatrix} 1 & d & e & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ give two 2-cells

$\begin{bmatrix} 1 & f & 0 & g \\ 0 & 0 & 1 & h \end{bmatrix}$ gives a 3-cell

$\begin{bmatrix} 1 & 0 & i & k \\ 0 & 1 & j & L \end{bmatrix}$ gives a 4-cell

We specify the attaching maps by rescaling and taking limits in the echelon forms. For example, the 1-cell $\begin{bmatrix} 0 & 1 & a & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1/a & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ attaches to

the 0-cell $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ as $a \to \infty$.

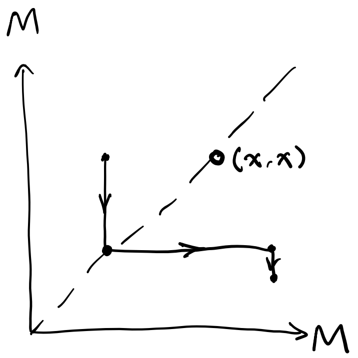
(b) (5 points) In the cellular chain complex $C_*^{\mathrm{CW}}(\mathrm{Gr}(2,4); \mathbb{Z}/2)$ from above, assume that all boundary maps are zero. Compute the (cellular) homology groups $H_n(\mathrm{Gr}(2,4); \mathbb{Z}/2)$.

Given the cellular chain complex

$$0 \leftarrow \mathbb{Z}/2 \xleftarrow{0} \mathbb{Z}/2 \xleftarrow{0} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \xleftarrow{0} \mathbb{Z}/2 \xleftarrow{0} \mathbb{Z}/2 \leftarrow 0 \leftarrow \cdots$$

we obtain $H_n(\mathrm{Gr}(2,4); \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & n = 0,1,3,4 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & n = 2 \\ 0 & \text{otherwise} \end{cases}$

4. (a) (5 points) Show that $\mathbb{R}$ is not the square of some topological space, i.e., there does not exist M such that $\mathbb{R}$ is homeomorphic to $M \times M$.



Suppose on the contrary that $\mathbb{R} \cong M \times M$. Since $\mathbb{R}$ is path connected, so is M. Now pick a point $x \in M$ and consider $M \times M - \{(x,x)\}$. Since M is path connected, this space is also path connected, but this is not the case for $\mathbb{R}$ with a point removed.

(b) (10 points) Let $\mathbb{F}$ be a field. Using the relative Künneth formula

$$H_n(X \times Y, A \times Y \cup X \times B; \mathbb{F}) \cong \bigoplus_{p+q=n} H_p(X, A; \mathbb{F}) \otimes_{\mathbb{F}} H_q(Y, B; \mathbb{F})$$

show that $\mathbb{R}^3$ is not the square of any space either.

Suppose that $\mathbb{R}^3 \cong M \times M$. Again, consider

$$X = Y = M \quad \text{and} \quad A = B = M - \{x\}$$

On one hand,

$$(X \times Y, A \times Y \cup X \times B) \cong (\mathbb{R}^3, \mathbb{R}^3 - pt) \simeq (\mathbb{R}^3, S^2)$$

has nontrivial homology only in dimensions 0 and 3 from the long exact sequence

$$\cdots \longrightarrow H_n(\mathbb{R}^3) \longrightarrow H_n(\mathbb{R}^3, S^2) \longrightarrow H_{n-1}(S^2) \longrightarrow \cdots$$

This then forces $(M, M - \{x\})$ on the other side of the Künneth isomorphism to have nontrivial homology in some dimension $p > 0$. By taking $q = p$, we conclude that $(\mathbb{R}^3, \mathbb{R}^3 - pt)$ has nontrivial homology in dimension $2p$, a contradiction.

5. Let R be a ring.

(a) (5 points) Write $\langle -, - \rangle$ for the evaluation pairing

$$C^i(X;R) \times C_i(X;R) \to R$$

Verify that the cup and cap products are adjoint operators with respect to $\langle -, - \rangle$ in the sense that, given $f \in C^m(X;R)$, $g \in C^n(X;R)$, and $x \in C_{m+n}(X;R)$, we have

$$\langle f \smile g, x \rangle = \langle f, g \frown x \rangle$$

This question caused some confusion during the exam and had to be modified near the end. Let's see where it goes wrong below.

By linearity, we may assume that the $(m+n)$-chain x is an $(m+n)$-simplex, written $[v_0, v_1, \ldots, v_{m+n}]$. It is then straightforward to verify by definition that

$$\langle f \smile g, x \rangle = (-1)^{mn} f([v_0, \ldots, v_m]) \cdot g([v_m, \ldots, v_{m+n}])$$

and

$$\begin{aligned} \langle f, g \frown x \rangle &= \langle f, g([v_0, \ldots, v_n]) \cdot [v_n, \ldots, v_{m+n}] \rangle \\ &= g([v_0, \ldots, v_n]) \cdot \langle f, [v_n, \ldots, v_{m+n}] \rangle \\ &= f([v_n, \ldots, v_{m+n}]) \cdot g([v_0, \ldots, v_n]) \end{aligned}$$

Thus the two sides do not agree at the cochain level. However, by switching f, g (and m, n) on the left-hand side, they do, up to a sign $(-1)^{mn}$ (not present in Hatcher). Passing to homology and cohomology classes, we then obtain

$$\begin{aligned} \langle [f] \smile [g], [x] \rangle &= \langle (-1)^{mn} [g] \smile [f], [x] \rangle \\ &= \langle [f], [g] \frown [x] \rangle. \end{aligned}$$

(b) (5 points) Using the relative cup product

$$H^m(X, A; R) \times H^n(X, B; R) \to H^{m+n}(X, A \cup B; R)$$

show that if X is the union of contractible open subsets A and B, then all cup products of positive-dimensional classes in $H^*(X; R)$ are zero. (In particular, this applies if X is a suspension, and its generalization for k-fold cup products should agree with your answer to part (b) of Question 2.)

In positive dimensions, (ordinary) cohomology $H^*(X;R)$ agrees with its reduced version $\widetilde{H}^*(X;R) \cong H^*(X, pt; R) \cong H^*(X, A; R)$ with A contractible. Suppose that $X = A \cup B$ with A and B both contractible. Via the relative cup product, we then see that any cup product of two (or more) positive-dimensional classes in $H^*(X;R)$ lands in $H^*(X, A\cup B; R) = H^*(X, X; R) = 0$.

In particular, a suspension SY (or its reduced version $\Sigma Y = S^1 \wedge Y$) is the union of the upper cone C_+Y and the lower cone C_-Y, both contractible (to their respective vertex through linear homotopies).

The projective space $\mathbb{RP}^m = \{[x_0 : \cdots : x_m] \mid x_i \in \mathbb{R}$ and at least one $x_i \neq 0\}$ is the union of $(m+1)$ contractible affine charts $\{(\frac{x_0}{x_i}, \ldots, \widehat{\frac{x_i}{x_i}}, \ldots, \frac{x_m}{x_i}) \mid x_i \neq 0\} \cong \mathbb{R}^m$, $i = 0, \ldots, m$.

(c) (10 points) Suppose that Y is a path-connected (based) space, M is a compact orientable manifold, and $f\colon S^1 \wedge Y \to M$ is a map which induces an isomorphism on homology with integer coefficients. Show that Y has the same homology as a sphere S^p for some p. (In fact, if Y is additionally a manifold itself, one can show that it must be (*homeomorphic* to) a sphere, rather than just a "homology sphere." You don't need to prove this here.)

Let $\dim M = n$ and $k = \min\{i > 0 \mid H_i(Y) \neq 0\}$.

Then $H_{k+1}(M) \cong \widetilde{H}_{k+1}(S^1 \wedge Y) \cong \widetilde{H}_k(Y) \neq 0$.

By the universal coefficient theorem, since $H_k(M) = 0$, we have $H^{k+1}(M) \neq 0$. Let x be a nonzero class in $H_{k+1}(M)$ and x^* be its dual class in $H^{k+1}(M)$.

On the other hand, by the Poincaré duality, $H^{n-k-1}(M) \xrightarrow{\frown [M]} H_{k+1}(M)$ is an isomorphism.

Let $y \in H^{n-k-1}(M)$ be such that $y \frown [M] = x$.

Then, by part (a), evaluating cohomology classes on homology classes gives $\langle x^* \smile y, [M] \rangle = \langle x^*, y \frown [M] \rangle = \langle x^*, x \rangle = 1$. If $k < n-1$, then y is in positive degree. However, this contradicts $x^* \smile y = 0$ from part (b). Therefore $k = n-1$ and the desired statement follows.

6. (a) (10 points) Using the Poincaré duality isomorphism

$$H^p(M;\mathbb{Z}/2) \to H_{n-p}(M,\partial M;\mathbb{Z}/2)$$

show that there is no compact manifold W with boundary $\partial W = \mathbb{RP}^{2k}$, where k is any positive integer.

Suppose on the contrary that such a W exists. For ease of notation, let us omit the coefficients $\mathbb{Z}/2$. Consider the following long exact sequence with known values of $H_*(\partial W)$ and $H_0(W)$:

$$\begin{array}{ccccccc} \cdots \to & H_{2k+1}(\partial W) & \to & H_{2k+1}(W) & \xrightarrow{j_{2k+1}} & H_{2k+1}(W,\partial W) & \xrightarrow{d_{2k+1}} \\ & 0 & & \begin{cases}\mathbb{Z}/2\\ 0\end{cases} & & \mathbb{Z}/2 & \end{array}$$

$$\begin{array}{ccccccc} \to & H_{2k}(\partial W) & \xrightarrow{i_{2k}} & H_{2k}(W) & \xrightarrow{j_{2k}} & H_{2k}(W,\partial W) & \xrightarrow{d_{2k}} \\ & \mathbb{Z}/2 & & (\mathbb{Z}/2)^s & & (\mathbb{Z}/2)^t & \end{array}$$

$$\begin{array}{ccccccc} \to & H_{2k-1}(\partial W) & \xrightarrow{i_{2k-1}} & H_{2k-1}(W) & \xrightarrow{j_{2k-1}} & H_{2k-1}(W,\partial W) & \xrightarrow{d_{2k-1}} \\ & \mathbb{Z}/2 & & (\mathbb{Z}/2)^m & & (\mathbb{Z}/2)^n & \end{array}$$

$$\begin{array}{ccc} \to & H_{2k-2}(\partial W) & \to \cdots \\ & \mathbb{Z}/2 & \end{array}$$

$$\begin{array}{cccccc} \to & H_2(\partial W) & \xrightarrow{i_2} & H_2(W) & \to & H_2(W,\partial W) \to \\ & \mathbb{Z}/2 & & (\mathbb{Z}/2)^n & & (\mathbb{Z}/2)^m \end{array}$$

$$\begin{array}{cccccc} \to & H_1(\partial W) & \xrightarrow{i_1} & H_1(W) & \to & H_1(W,\partial W) \to \\ & \mathbb{Z}/2 & & (\mathbb{Z}/2)^t & & (\mathbb{Z}/2)^s \end{array}$$

$$\begin{array}{cccccc} \to & H_0(\partial W) & \xrightarrow{i_0} & H_0(W) & \to & H_0(W,\partial W) \to 0 \\ & \mathbb{Z}/2 & & \mathbb{Z}/2 & & \begin{cases}\mathbb{Z}/2\\ 0\end{cases} \end{array}$$

By the Poincaré duality, $H_{2k+1}(W,\partial W) \cong H^{2k+1}(W,\partial W)^* \cong H_0(W)^* \cong \mathbb{Z}/2$. Since j_{2k+1} is injective, $H_{2k+1}(W) \cong H_0(W,\partial W)$ can only be either $\mathbb{Z}/2$ or 0. We carry out a reduction step by cases as follows.

Case 1 $H_{2k+1}(W) \cong \mathbb{Z}/2$. Then j_{2k+1} is an isomorphism, which implies $d_{2k+1}=0$ and hence i_{2k} is injective. Suppose that $H_{2k}(W) \cong (\mathbb{Z}/2)^s$ and $H_{2k}(W, \partial W) \cong (\mathbb{Z}/2)^t$. Then, since $\ker j_{2k} = \operatorname{im} i_{2k} \cong \mathbb{Z}/2$, we get $t \geq s-1$. On the other hand, $\mathbb{Z}/2 \supset \operatorname{im} d_{2k} \cong (\mathbb{Z}/2)^t / \ker d_{2k} \cong (\mathbb{Z}/2)^t / \operatorname{im} j_{2k}$ $\cong (\mathbb{Z}/2)^{t-s+1}$, and so $1 \geq t-s+1$ or $s \geq t$. Thus $t = s-1$ or $t=s$.

- When $t=s-1$, j_{2k} is surjective and so $d_{2k} = 0 = d_{2k+1}$. On the other hand, by the Poincaré duality, $H_1(W, \partial W) \cong (\mathbb{Z}/2)^s$ and $H_1(W) \cong (\mathbb{Z}/2)^t = (\mathbb{Z}/2)^{s-1}$. A similar diagram chasing yields $i_1 = 0 = i_0$. This way, to find a contradiction from the long exact sequence, we have reduced the portion between $d_{2k+1}=0$ and $i_0=0$ to that between $d_{2k}=0$ and $i_1=0$.

- When $t=s$, d_{2k} is surjective, which implies $i_{2k-1}=0$ and hence j_{2k-1} is injective. Suppose that $H_{2k-1}(W) \cong H_2(W, \partial W) \cong (\mathbb{Z}/2)^m$ and $H_{2k-1}(W, \partial W) \cong H_2(W) \cong (\mathbb{Z}/2)^n$. Then $n=m$ or $n=m+1$.

  If $n=m$, then $d_{2k-1}=0$ and $i_2=0$. Thus we have reduced the portion of the long exact sequence between $d_{2k+1}=0$ and $i_0=0$ to that between $d_{2k-1}=0$ and $i_2=0$.

  If $n=m+1$, then d_{2k-1} is surjective and i_2 is an isomorphism. Thus we have reduced the portion between the surjection d_{2k} and the isomorphism i_1 to that between the surjection d_{2k-1} and the isomorphism i_2.

Case 2 $H_{2k+1}(W) = 0$. Then d_{2k+1} is surjective and i_0 is an isomorphism. Thus we have reduced the portion between the surjection d_{2k+1} and the isomorphism i_0 to that between the surjection d_{2k} and the isomorphism i_1 in Case 1 when $t=s$.

Inductively, by the two cases of reduction above, it remains to examine the following middle portion of the long exact sequence for 3 possibilities:

$$
\begin{tikzcd}
\underset{\mathbb{Z}/2}{H_{k+2}(\partial W)} \arrow[r, "i_{k+2}"] & H_{k+2}(W) \arrow[r] & H_{k+2}(W,\partial W) \arrow[dll, rounded corners, "d_{k+2}"] \\
\underset{\mathbb{Z}/2}{H_{k+1}(\partial W)} \arrow[r, "i_{k+1}"] & \underset{(\mathbb{Z}/2)^s}{H_{k+1}(W)} \arrow[r, "j_{k+1}"] & \underset{(\mathbb{Z}/2)^t}{H_{k+1}(W,\partial W)} \arrow[dll, rounded corners, "d_{k+1}"] \\
\underset{\mathbb{Z}/2}{H_k(\partial W)} \arrow[r, "i_k"] & \underset{(\mathbb{Z}/2)^t}{H_k(W)} \arrow[r, "j_k"] & \underset{(\mathbb{Z}/2)^s}{H_k(W,\partial W)} \arrow[dll, rounded corners] \\
\underset{\mathbb{Z}/2}{H_{k-1}(\partial W)} \arrow[r, "i_{k-1}"] & H_{k-1}(W) \arrow[r] & H_{k-1}(W,\partial W)
\end{tikzcd}
$$

1st Possibility $d_{k+1} = 0$ and $i_k = 0$. This contradicts $H_k(\partial W) \cong \mathbb{Z}/2$.

2nd Possibility $d_{k+2} = 0$ and $i_{k-1} = 0$ and $t = s$. Then d_{k+1} is surjective and i_k is an isomorphism. Again, this contradicts $H_k(\partial W) \cong \mathbb{Z}/2$.

3rd Possibility d_{k+2} is surjective and i_{k-1} is an isomorphism. Then $t=s$ or $t=s+1$. If $t=s$, then $d_{k+1}=0$ and $i_k=0$, a contradiction. If $t=s+1$, then d_{k+1} is surjective and i_k is an isomorphism, another contradiction.

(b) (5 points) Find such a manifold W with $\partial W = \mathbb{RP}^{2k+1}$. (Hint: Suitably view $\mathbb{RP}^{2k+1}$ as a fiber bundle over $\mathbb{CP}^k$.)

Define $f: \mathbb{RP}^{2k+1} \longrightarrow \mathbb{CP}^k$ by sending each $x \in \mathbb{RP}^{2k+1}$, viewed as a one-dimensional subspace of $\mathbb{R}^{2k+2} \cong \mathbb{C}^{k+1}$ spanned by $v = (v_1, \dots, v_{2k+2}) = (v_1 + iv_2, \dots, v_{2k+1} + iv_{2k+2})$, to the complex plane (or subspace of complex dimension one) containing it and spanned by v and $iv = (iv_1 - v_2, \dots, iv_{2k+1} - v_{2k+2})$. This gives $\mathbb{RP}^{2k+1}$ the structure of a circle bundle over $\mathbb{CP}^k$ whose fiber $S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$. Let W be the disc bundle obtained by filling S^1 to $D^2 = \{ z \in \mathbb{C} \mid |z| \leq 1 \}$.