

MAT7064, Topics in Geometry and Topology

Assignment 8

Due in-class on Friday, November 29

One way that spectral sequences arise is through what is called an *exact couple*. It arises algebraically in the following situation (which is a simplification of one we would actually use).

Suppose that we have abelian groups D and E , and maps $i : D \rightarrow D$, $j : D \rightarrow E$, and $k : E \rightarrow D$ such that

$$\cdots \rightarrow D \rightarrow D \rightarrow E \rightarrow D \rightarrow D \rightarrow E \rightarrow \cdots$$

is exact. We often write this in a triangle:

$$\begin{array}{ccc} D & \xrightarrow{i} & D \\ & \swarrow k & \searrow j \\ & E & \end{array}$$

1. Show that the map $d = jk$ satisfies $d^2 = 0$, and so we get a homology group $H(E) = \ker(d)/\text{im}(d)$.
2. Let $i(D) = \text{im}(i)$. Show that the map k induces a well-defined map $k' : H(E) \rightarrow i(D)$.
3. Show that the description $j'(x) = ji^{-1}(x)$ gives us a well-defined map $i(D) \rightarrow H(E)$.
4. Show that these maps give us a new exact couple:

$$\begin{array}{ccc} i(D) & \xrightarrow{i} & i(D) \\ & \swarrow k' & \searrow j' \\ & H(E) & \end{array}$$

In other words, show that these three maps give a new long exact sequence. This is called the *derived* exact couple, and this procedure can be iterated again, and again, and again...