Replication and Application of Topological Music Analysis to Baroque Concertos

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Abstract

This work follows and implements the framework for topological music analysis proposed by Alcalá-Alvarez and Padilla-Longoria [1], applying persistent homology to the study of harmonic structures in Western tonal music. Our primary goal is not to introduce new topological constructions, but to carefully reproduce and systematize the existing methodology, and to test its descriptive power in a conservative setting. We extract sequences of vertical events from musical fragments and encode them in several ways. For each encoding, we compute persistent homology using Vietoris-Rips filtrations and compare the resulting persistence diagrams via bottleneck distance.

In contrast to previous studies that focus on stylistically distinct repertoires, we restrict attention to four Baroque concertos sharing closely related harmonic and stylistic features. This controlled setting allows us to examine whether the established TMA pipeline can capture subtle differences among highly similar works. The resulting comparisons are consistent with traditional tonal analysis, while also highlighting how different data encodings emphasize different harmonic aspects.

1 Introduction

This work will present several ways to process the data extracted from digital music scores, and apply some methods from topological data analysis to analyze the processed data. Subsequently, these methods will be applied to compare some baroque concertos. The main motivation comes from the work of Alcala-Padilla et al. [1], in which they applied persistent homology to analyze and compare several music pieces from different styles. So in this work, we will follow their procedure step by step, and try to use the same methods to make a more conservative test by selecting four baroque concertos sharing similar stylistic features, and see whether our method can capture subtle differences among them.

There have been several perspectives to analyze music all over the world, such as traditional music tonal theory, Chinese pentatonic scale, Indian raga, and Western harmony theory, and so on. Undeniably, these theories have provided us with a deep understanding of music from different cultural backgrounds and have done a great job in analyzing and composing music in their respective contexts. However, when it comes to describing musical objects and phenomena found in various styles, musicians and musicologists, a single framework may not be sufficient and consistent. Thus if there is a more general framework to analyze music, works will be more clear and easy when dealing with music from different styles even though some details may be lost.

In recent years, topological data analysis (TDA) has emerged as a powerful tool for analyzing complex data sets by capturing their underlying topological features [1]. Scores of music compositions can be mapped into high-dimensional spaces in many ways, where different ways focus on different aspects of music. By applying TDA techniques, especially persistent homology, we can get some information by comparing different music pieces, which may reveal new insights into the structure and relationships between music, and will be fully established in the following sections.

First, we we apply persistent homology to different sets of points formed from pitch with or without rhythm and onset time, which are called vertical events. Next, we propose two ways to construct sequences of simplicial complex from chord sequences. One of this consrtuction has been proposed as a way to regard a chord as a simplex on vertices corresponding to pitches or pitch classes [2], assuming chords consisting of n pitch classes as simplices on n vertices. We plot as barcodes and persistence diagrams the persistent homology of these constructions, and compare different baroque concertos based on these topological features, using bottleneck distance. Finally, we summarize the results and discuss possible future directions.

2 Definitions and Methods

We consider music scores written in stave notation, which can be digitally represented in formats such as MusicXML or MIDI. We define **vertical events** as tuples containing information of pitches (or pitch classes), possibly together with some other features like its onset, duration (rhythm), dynamics (loudness) or timbre (instrument). In this work, we mainly focus on pitches, with or without onset time and duration of vertical events.

A vertical event consisting of only pitches is called a **chord**, and a chord formed by n different pitch classes is called an **n-chord**. A sequence of vertical events is called a **music fragment**. For a music fragment $\mathcal{M} = \{e_0, \ldots, e_n\}$, the interval events $[e_i, e_j]$ is the sequence of vertical events from e_i to e_j.

This segmentation only depends on the sequential order, since it is indexed by their order of appearance in the score. And hence it allows us to concentrate on harmonic changes, when necessary, take into account time information. We always take indexes in to consideration, because oen chord may appear multiple times in a music fragment. So $\mathcal{M}$ is actually in the form $\{(e_0, 0), (e_1, 1), \ldots, (e_n, n)\}$. We will omit the indexes when there is no ambiguity. This notation helps us to consider harmonic progressions independently of rythmic values involved. Simplices corresponding to repeated chords will only appear once in the simplicial complex. Also, when being mapped into $\mathbb{R}^n$, repeated chords will be mapped to the same point.

Given a music fragment $\mathcal{M} = \{e_0, \ldots, e_n\}$, we define $\mathcal{A}(\mathcal{M}) = \{a_0, a_1, \ldots, a_n\}$ as the ordered set where a_i corresponds to the i-th vertical event e_i in $\mathcal{M}$. Chords in the set $\mathcal{A}(\mathcal{M})$ can be represented as points in $\mathbb{R}^n$ in several ways, depending on the features we want to focus on: by their common name like C major or A minor etc, by their pitch classes, or by their intervals defined in classical post-tonal theory. Later, we will propose six ways to construct $\mathcal{A}(\mathcal{M})$ from a music fragment $\mathcal{M}$, and apply persistent homology.

Now we consider pitches defined in the twelve-tone equal temperament system. Their corresponding pitch calsses can be represented as integers modulo 12, where C=$\bar{0}$, C$\sharp$=$\bar{1}$, D=$\bar{2}$, D$\sharp$=$\bar{3}$, E=$\bar{4}$, F=$\bar{5}$, F$\sharp$=$\bar{6}$, G=$\bar{7}$, G$\sharp$=$\bar{8}$, A=$\bar{9}$, A$\sharp$=$\bar{10}$, B=$\bar{11}$. This representation allows us to ignore octave differences and focus on the relationships between pitch classes.

Since we only care about things about harmonic, before we start, we delete all staves containing unpitched percussion instruments, and only keep staves containing pitched instruments. In order to unify the time, we express all onset times and durations in terms of quarter note (crotchet). This standardization will not restrict ourselves to the notes only with quarter note duration, instead, the quarter note merely serves as time unit, we can still use miliseconds or ticks to express onset times and durations. Additionally, the choice of time unit will not affect the data being mapped into $\mathbb{R}^n$, up to a scaling factor. Nevertheless, homeomorphic data encodings lead to similar shape features under TDA analysis, since homology is a topological invariant.

Here is an example under the above setting.

Example 2.1. Let us consider the first measure of the first movement of Bach's Brandenburg Concerto No. 5, BWV 1050, one of the most famous baroque concertos, as shown in Figure 1.

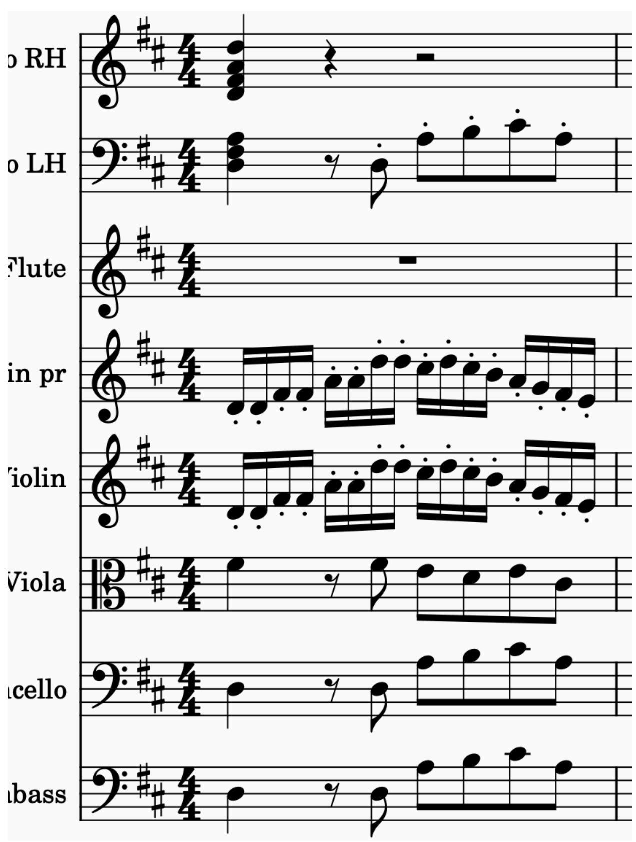

Figure 1: The first measure of Brandenburg Concerto No. 5 in D, BWV 1050: I Allegro.

Below we explicitly list the 16 vertical events appearing in this measure, as shown in Figure 1, along with their onset times and durations (in quarter note units), with the form

$$(\{\text{pitch classes}\}, \text{duration}, \text{onset}).$$

That is, from the score in Figure 1, we have the following musical fragment

$$\begin{aligned}\mathcal{M} = \{&(\{\bar{2},\bar{6},\bar{9}\},0.25,0),(\{\bar{2},\bar{6},\bar{9}\},0.25,0.25),(\{\bar{2},\bar{6},\bar{9}\},0.25,0.5),(\{\bar{2},\bar{6},\bar{9}\},0.25,0.75),\\ &(\{\bar{9}\},0.25,1),(\{\bar{9}\},0.25,1.25),(\{\bar{2},\bar{6}\},0.25,1.5),(\{\bar{2},\bar{6}\},0.25,1.75),\\ &(\{\bar{1},\bar{4},\bar{9}\},0.25,2),(\{\bar{2},\bar{4},\bar{9}\},0.25,2.25),(\{\bar{1},\bar{2},\bar{11}\},0.25,2.5),(\{\bar{2},\bar{11}\},0.25,2.75),\\ &(\{\bar{1},\bar{4},\bar{9}\},0.25,3),(\{\bar{1},\bar{4},\bar{7}\},0.25,3.25),(\{\bar{1},\bar{9},\bar{6}\},0.25,3.5),(\{\bar{1},\bar{9},\bar{4}\},0.25,3.75)\}.\end{aligned}$$

There is no doubt that we can get different musical fragments from the same score, by modifying vertical events to include or exclude onset times and durations. An notice that an event occurs only when there is a change in notes sounding simultaneously. It is common that a pitch may last longer than the duration of a vertical event, once it happens, same pitch will appear in the following vertical events until it ends, to fulfill its actual duration. For example, as listed above, in the ninth and tenth vertical events that start at time 2 and 2.25 respectively, pitch class $\bar{9}$ (A) and $\bar{4}$ (E) still sound. Periodic decimal temporal values will be truncated if it appears.

Now from the above fragment $\mathcal{M}$, we get

$$\begin{aligned}\mathcal{M} = \{&(\bar{2},\bar{6},\bar{9})_1,(\bar{2},\bar{6},\bar{9})_2,(\bar{2},\bar{6},\bar{9})_3,(\bar{2},\bar{6},\bar{9})_4,\\ &(\bar{9})_5,(\bar{9})_6,(\bar{2},\bar{6})_7,(\bar{2},\bar{6})_8,\\ &(\bar{1},\bar{4},\bar{9})_9,(\bar{2},\bar{4},\bar{9})_{10},(\bar{1},\bar{2},\bar{11})_{11},(\bar{2},\bar{11})_{12},\\ &(\bar{1},\bar{4},\bar{9})_{13},(\bar{1},\bar{4},\bar{7})_{14},(\bar{1},\bar{9},\bar{6})_{15},(\bar{1},\bar{9},\bar{4})_{16}\}.\end{aligned}$$

which is one kind of musical fragment we will consider later. In the following text, we will omit the indexes of chords and events. Later we will encode these events in different ways, for example

$$(\text{interval vector of chord}, \text{duration}, \text{onset})$$

which will be mapped into $\mathbb{R}^n$ in the form

$$(\text{chords as vector in some Euclidean space}, \text{duration}, \text{onset})$$

or simply vectors representing chords only, without duration and onset time. In order to apply persistent homology, all those representing chords must belong to the same $\mathbb{R}^n$. Now, here are six ways to construct $\mathcal{A}(\mathcal{M})$ from a music fragment $\mathcal{M}$.

2.1 General Strategy

The first time we get a music score, we extract all vertical events from it in normal form, just as the example shows: all pitches together with their onset times and durations, both in quarter note units. Similarly, we can also consider those tuples in which chords are encoded as interval vectors, where interval vector is a classical way to represent pitch class sets in post-tonal theory, for example, the interval vector of major triad $\{\bar{0}, \bar{4}, \bar{7}\}$ is (0,0,1,1,1,0), since it contains one minor third (3 semitones), one major third (4 semitones) and one perfect fifth (7 semitones). Despite the fact that these representations indeed capture some harmonic features of chords, while they also lose some information like voicing and inversion of chords. Thus, we may model chord connections rather than chords themselves to take them into consideration, which will lead to other embbedings and associated spaces. All such events will be analyzed both with or without onset times and durations.

The most important part of analysis involves TDA techniques, especially persistent homology. After we map musical events into some Euclidean space $\mathbb{R}^n$, we can construct a filtration of simplicial complexes from the point cloud formed by these points, using Vietoris-Rips complex or Cech complex, and compute their persistent homology under Euclidean metric. While we will also compute persistent homology of simplicial complexes directly associated with event intevals, without considering a metric among data points.

All computations are done using Python programming language. We use Music21 library to extract musical events from digital music scores in MIDI format. And we use MoguTDA library to compute persistent homology and plot barcodes and persistence diagrams, and calculate bottleneck distances between persistence diagrams. Some standard library like NumPy and Matplotlib are also used for data processing and visualization.

I download digital music scores in MIDI format from MuseScore, which is a popular platform for sharing music scores. It can also identify music scores in PDF format and convert them into MIDI format, if such scores are not available on the platform.

2.2 Persistent Homology on Various Musical Data Mapping

In this section, we will describe six ways to construct the ordered set $\mathcal{A}(\mathcal{M})$ from a music fragment $\mathcal{M}$, and apply persistent homology to each of them. We number the mapped data sets from I to VI, and refer to those sets as data mappings.

In the first two mappings I and II we consider the points that stand for vertical events in the music fragment $\mathcal{M}$ and incorporate onset times and durations: all events will be translated as a tuple of pitch or interval classes, together with a rhythm value and an onset time value, both in quarter note units, just as the example shows. The rest of the mappings III, IV, V and VI only consider pitches or intervals, without onset times and durations. They only focus on harmonic aspects of $\mathcal{M}$. Once we finish generating those data sets, we apply persistent homology to each of them, using Vietoris-Rips complex under Euclidean metric, and plot their barcodes and persistence diagrams. Sequentially, we compare different baroque concertos based on these topological features, using bottleneck distance, which is a standard way to measure the difference between two persistence diagrams. Therefore, we are able to present in dendrograms the distances among different music pieces with the same data mapping. And hence we establish a notion of closseness among different music pieces, from the perspective of topological data analysis.

For better presentation and visualization, we will mannually process the first movement of BWV 1050: I Allegro, which has been shown in the example above, and we will exhibit the results of persistent homology computations of BWV 1048: I Allegro (the first four movements) as an example in each data mapping. We use different colors in persistence diagrams to represent different homology dimensions. As for plotting barcodes, we use teal for data mapping which include onset times and durations, and use purple for data mapping without onset times and durations. Due to computational time, we only compute persistent homology up to dimension 3.

2.2.1 Data Mapping I: Pitches with Onset Times and Durations

In this data mapping, each vertical event is mapped to a point in $\mathbb{R}^{14}$ whose first 12 coordinates represent the pitch classes, the 13th coordinate represents the duration, and the 14th coordinate represents the onset time. In these vectors, coordinates representing pitch classes of a chord are represented by integers from 12 to 23, according to their ordering in the chords normal form. Such choice allows us to embed any n-chord with $n \leq 12$ into $\mathbb{R}^{14}$. The rest two coordinates are simply the duration and onset time, given as decimal numbers in quarter note units.

For example, three different vertical events in Figure 1 are mapped as follows:

$$(\{\bar{2}, \bar{6}, \bar{9}\}, 0.25, 0) \mapsto (14, 18, 21, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.25, 0)$$
$$(\{\bar{1}, \bar{4}, \bar{9}\}, 0.25, 2) \mapsto (13, 16, 21, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.25, 2)$$
$$(\{\bar{2}, \bar{11}\}, 0.25, 2.75) \mapsto (14, 23, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.25, 2.75)$$

Applying this data mapping to the first four measures of BWV 1048: I Allegro, we get persistence and barcodes diagrams as shown in Figure 2 and Figure 3 respectively. Descrbing events as vectors recording pitches, durations and onset times, this mapping gives us a way to have a glimps into their general distribution over time, and also identify the presence of distinguished harmonic regions. In this way, we can have a general impression of harmonic-rhythmic structure of the fragment.

2.2.2 Data Mapping II: Interval Vectors with Onset Times and Durations

Similarly, we consider each event as a point in $\mathbb{R}^8$ whose first 6 coordinates represent the interval vector of the chord, the 7th coordinate represents the duration, and the 8th coordinate represents the onset time. This mapping reflects similarity in the chord structures present in each event, together with their distribution in time and rhythm. For example, three different vertical events in

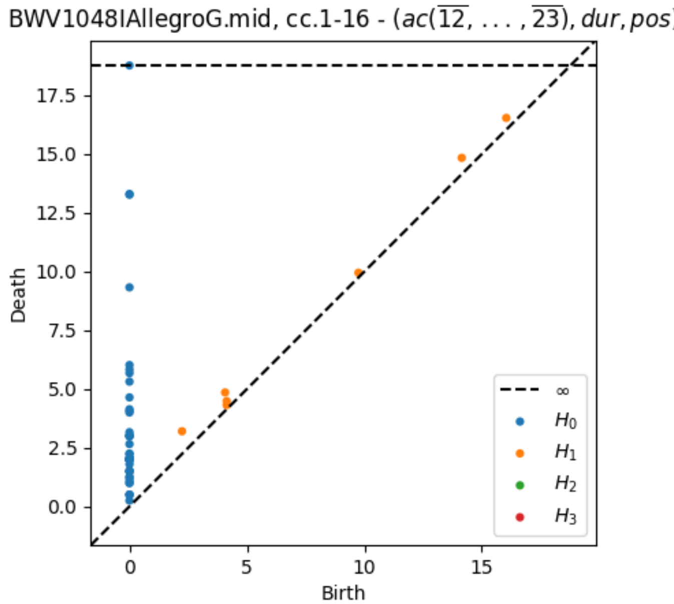

Figure 2: Persistence diagram of BWV 1048: I Allegro under Data Mapping I.

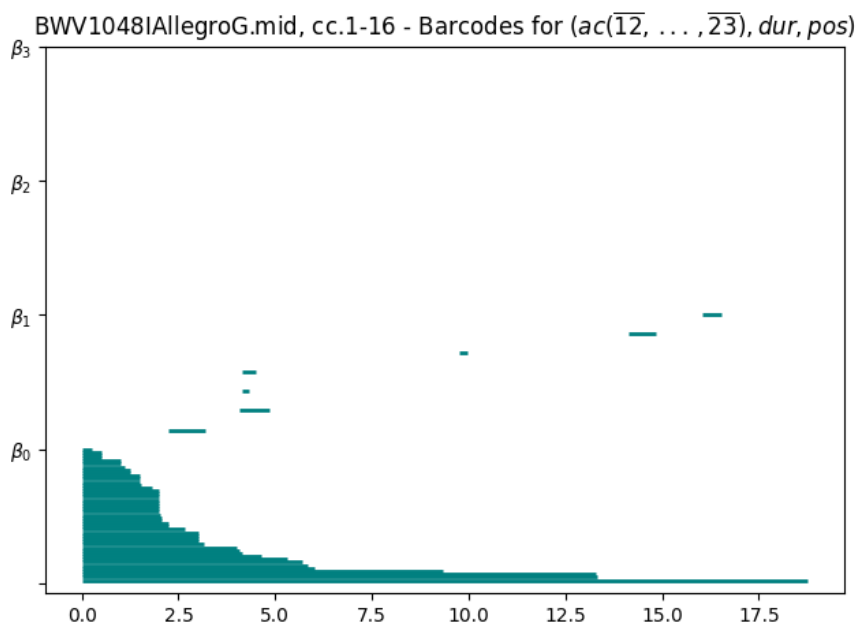

Figure 3: Barcode of BWV 1048: I Allegro under Data Mapping I.

Figure 1 are mapped as follows:

$$(\{\bar{2},\bar{6},\bar{9}\},0.25,0) \mapsto (0,0,1,1,1,0,0.25,0)$$
$$(\{\bar{1},\bar{4},\bar{9}\},0.25,2) \mapsto (0,0,1,1,1,0,0.25,2)$$
$$(\{\bar{2},\bar{4},\bar{9}\},0.25,2.25) \mapsto (0,1,1,0,1,0,0.25,2.25)$$

The resulting persistence and barcodes diagrams of the first four measures of BWV 1048: I Allegro are shown in Figure 4 and Figure 5 respectively. This data mapping together with mapping III

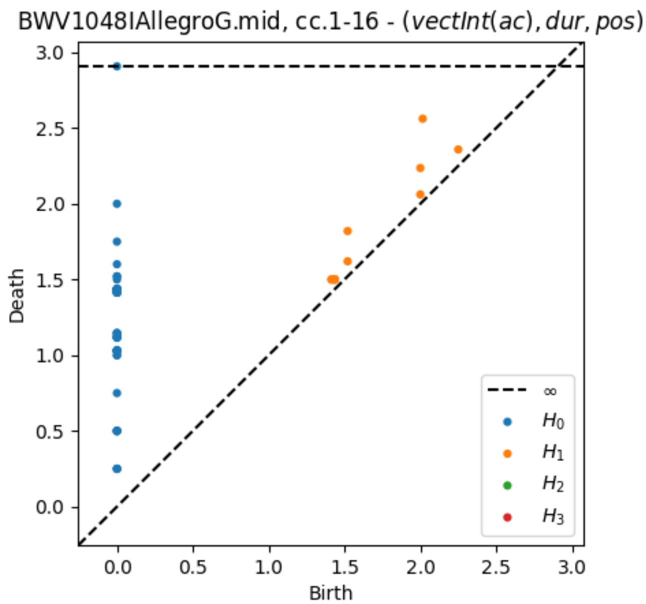

Figure 4: Persistence diagram of BWV 1048: I Allegro under Data Mapping II.

below, focuses more on types of chords or different chord structures presented in the fragment, as they deal with interval vectors rather than pitches themselves.

Parallel to the above, we also work on sets obtained only from pitch data. We still focus on harmony by considering data points only containing pitches information, without onset times and durations. We use four ways to analyze the same data, as described below.

2.2.3 Data Mapping III: Interval Vectors Only

Dropping the last two coordinates in Data Mapping II, we consider each event as a point in $\mathbb{R}^6$ whose coordinates represent the interval vector of the chord. Following the same example, we get

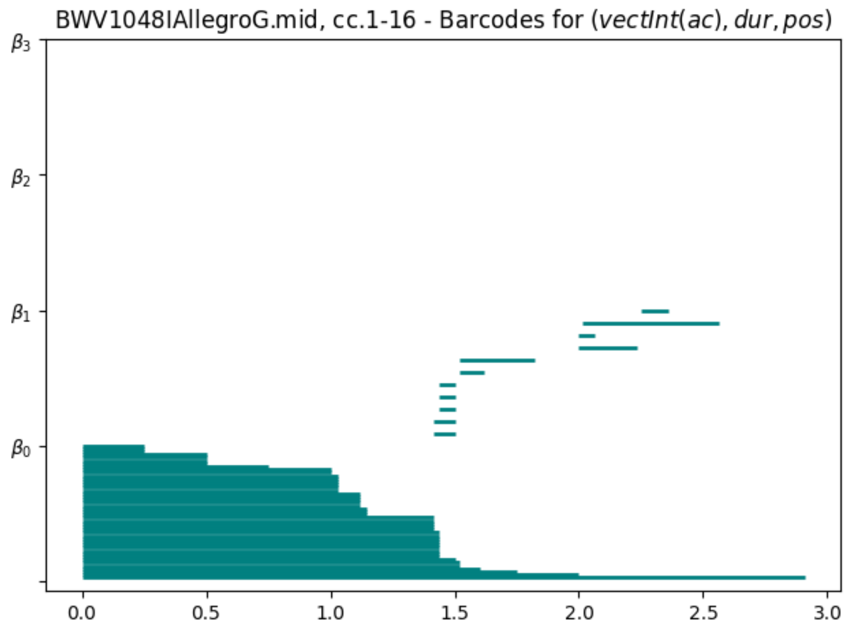

Figure 5: Barcode of BWV 1048: I Allegro under Data Mapping II.

the ordered set $\mathcal{A}_{int.vect.}(M)$ as follows:

$$\begin{aligned}
\{&(0,0,1,1,1,0),(0,0,1,1,1,0),(0,0,1,1,1,0),(0,0,1,1,1,0)\\
&(0,0,0,0,0,0),(0,0,0,0,0,0),(0,0,0,1,0,0),(0,0,0,1,0,0)\\
&(0,0,1,1,1,0),(0,1,0,0,2,0),(1,1,1,0,0,0),(0,0,1,0,0,0)\\
&(0,0,1,1,1,0),(0,0,2,0,0,1),(0,0,1,1,1,0)(0,0,1,1,1,0)\}
\end{aligned}$$

In Figure 6 and Figure 7, we show the persistence diagram and barcode of the first four measures of BWV 1048: I Allegro under this data mapping respectively. This is a coarsest way to analyze harmonic structures in a music fragment, since it is purely based on interval content modulo inversions without considering any other information. This implies a simpler shape of the data set, which may lead to less homological features being presented. Hence, it is expected that the persistent homology will be less informative compared to other data mappings. But somehow they summerize diagrams obtained from other data mappings, since images of events under any of them can be projected onto the set of their interval vectors.

2.2.4 Data Mapping IV: Presence of Pitches

This mapping maps chords as vectors in $\{0,1\}^{12} \subset I^{12} \subset \mathbb{R}^{12}$, where $I = [0,1]$ is given in the following way:

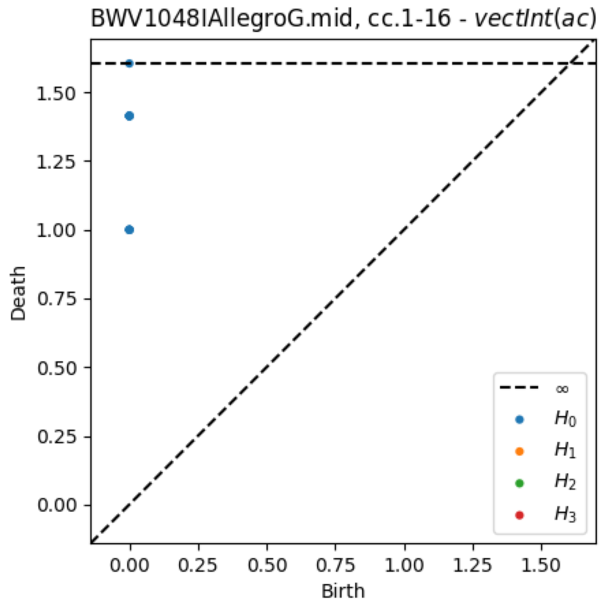

Figure 6: Persistence diagram of BWV 1048: I Allegro under Data Mapping III.

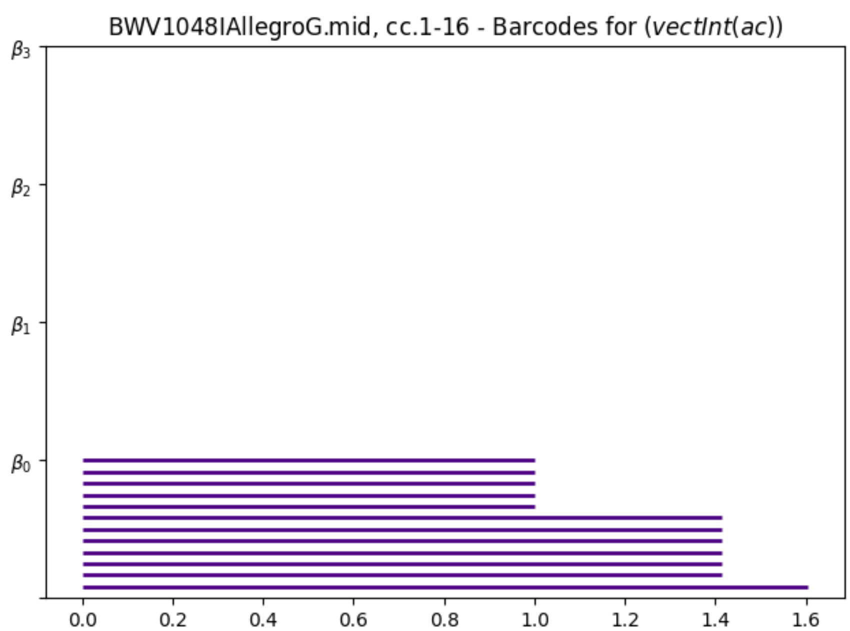

Figure 7: Barcode of BWV 1048: I Allegro under Data Mapping III.

Given a chord a in normal form, we define $a_{I^{12}} = (r_0, r_1, \dots, r_{11})$ where

$$r_i = \begin{cases} 1, & \text{if } \bar{i} \in a \\ 0, & \text{if } \bar{i} \notin a \end{cases}$$

For example, three different vertical events in Figure 1 are mapped as follows:

$$(\{\bar{2}, \bar{6}, \bar{9}\}) \mapsto (0,0,1,0,0,0,1,0,0,1,0,0)$$
$$(\{\bar{1}, \bar{4}, \bar{9}\}) \mapsto (0,1,0,0,1,0,0,0,0,1,0,0)$$
$$(\{\bar{2}, \bar{4}, \bar{9}\}) \mapsto (0,0,1,0,1,0,0,0,0,1,0,0)$$

This association gives teh sequence

$$\mathcal{A}_{I^{12}}(\mathcal{M}) = \{a_{i_{I^{12}}} | a_i \in \mathcal{M}\}$$

The resulting persistence and barcodes diagrams of the first four measures of BWV 1048: I Allegro are shown in Figure 8 and Figure 9 respectively. In this setting, each dimention represents the

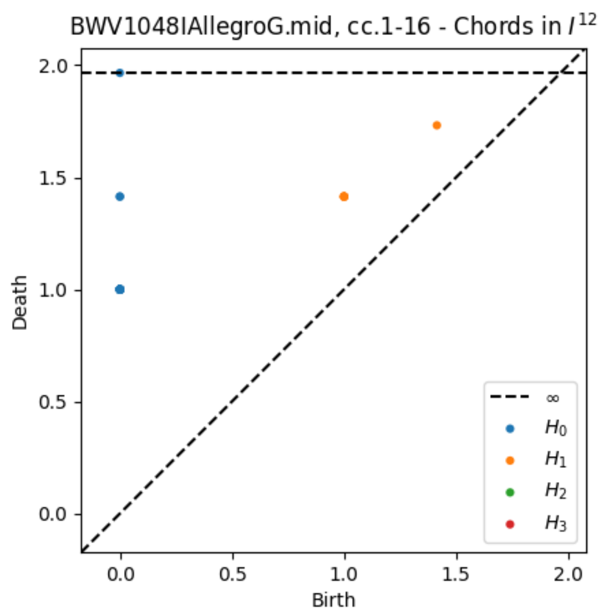

Figure 8: Persistence diagram of BWV 1048: I Allegro under Data Mapping IV.

presence or absence of a particular pitch class in a chord. So the images of two events will be close if they share many pitch classes, and far apart if they have few pitch classes in common. Moreprecisely, a given chord a contains k different pitch classes, if and only if $\|a_{I^{12}}\| = \sqrt{k}$. As a corollary, for chords a and b, we have $\|a_{I^{12}} - b_{I^{12}}\| = \sqrt{k}$, if and only if a and b differ by k pitch classes. And hence we can measuer how close chords are among themselves in terms of common

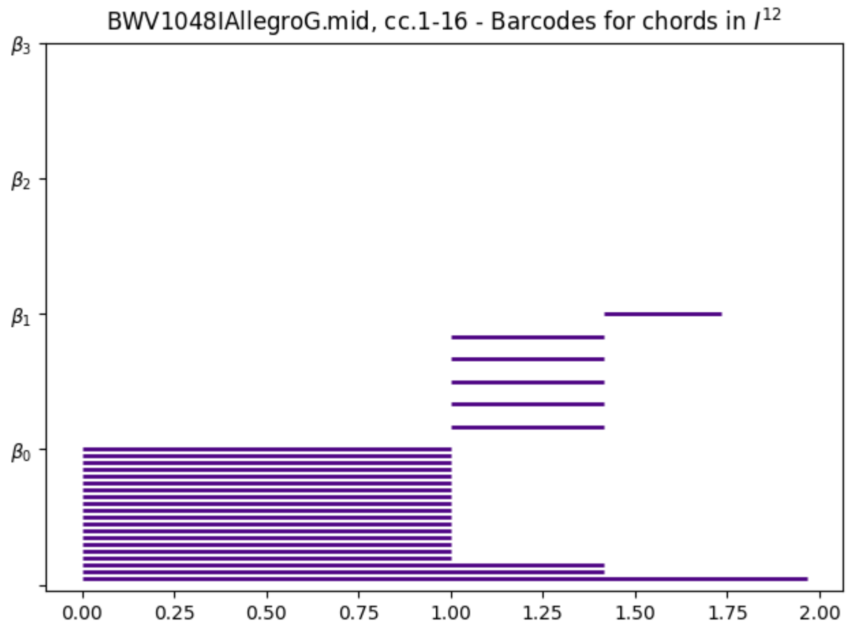

Figure 9: Barcode of BWV 1048: I Allegro under Data Mapping IV.

pitch class.

This mapping yields homological features in higher dimensions than mappings III, V and VI. Also, these features persist only during specific intervals, determined by the squre root of integers. So bar in barcodes appear to form blocks.

2.2.5 Data Mapping V: Pitches Only

Projecting the data from mapping I on its first 12 coordinates, we consider each event as a point in $\mathbb{R}^{12}$. In this case, n-chords are mapped to tuples with non-zero integer values between 12 and 23 in the first n coordinates, and 0 in the rest coordinates. As a consequence, a chord is a n-chord if and only if its image belongs to the subspace spanned by the first n canonical basis vectors of $\mathbb{R}^{12}$. Thus, through this mapping, samples produce similar diagrams if and only if their events are similar in pitch and number of harmonic voices.

In our example, the chord $(\{\bar{2}, \bar{6}, \bar{9}\})$ is mapped as follows:

$$(\{\bar{2}, \bar{6}, \bar{9}\}) \mapsto (14, 18, 21, 0, 0, 0, 0, 0, 0, 0, 0, 0).$$

This mapping and mapping I are the only two mappings that are sensitive to transposition of the fragment by a given interval. The bottleneck distance between persistence diagrams for this data mapping of a fragment and its transposition will not be always 0. Through this mapping, we capture closseness of chords in terms of both pitch content.

The resulting persistence and barcodes diagrams of the first four measures of BWV 1048: I Allegro are shown in Figure 10 and Figure 11 respectively.

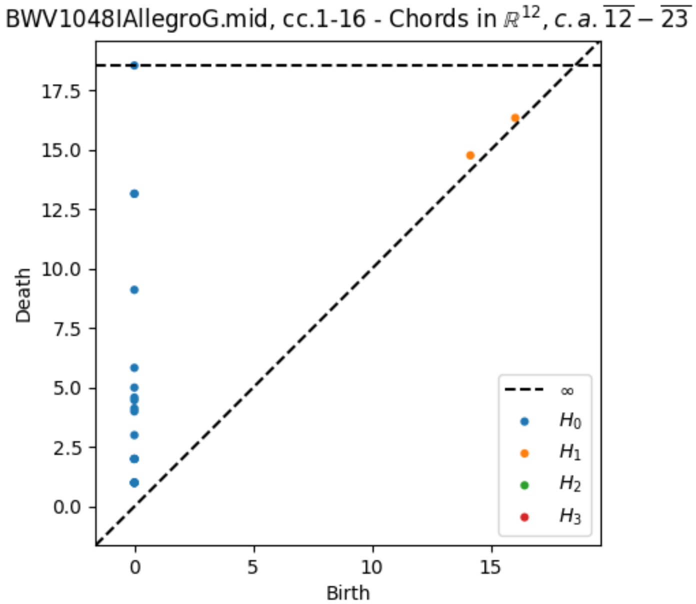

Figure 10: Persistence diagram of BWV 1048: I Allegro under Data Mapping V.

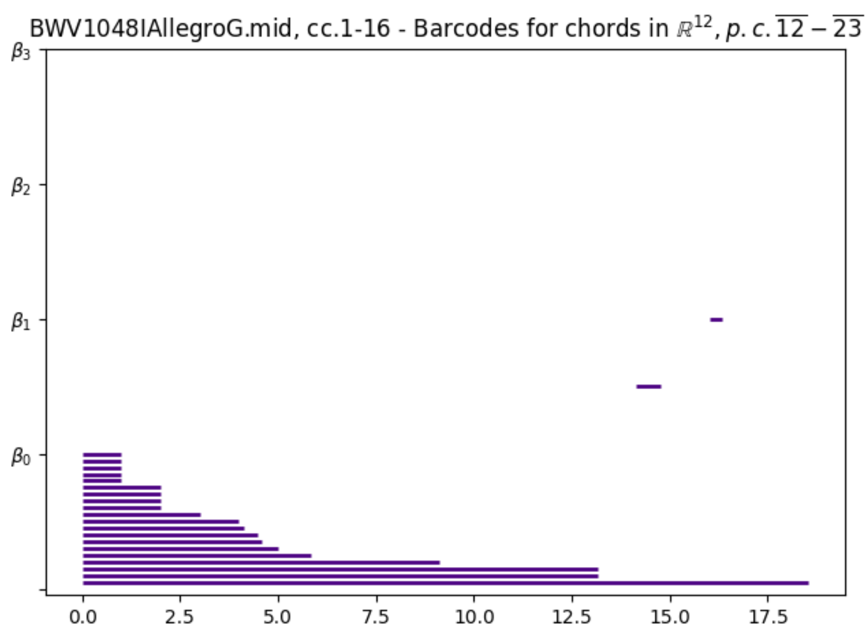

Figure 11: Barcode of BWV 1048: I Allegro under Data Mapping V.

2.2.6 Data Mapping VI: Pitches and Next Pitches

Finally, we consider not only the pitches, but also the intervals between two consecutive chords. We propose the association $a \mapsto a_{p,int} = (r_0, r_1, \ldots, r_{11}, d_0, d_1, \ldots, d_{11}) \in \mathbb{R}^{12}$, where $0 \leq r_i < 12$ stands for the interval class between pitch class $\bar{i}$ and the next pitch class in the normal form of chord a, if $\bar{i}$ is in a, and $r_i = 0$ otherwise. Explicitly, that is

$$r_i = \begin{cases} k & \text{k is the next interval in a} \\ 0, & \text{if } \bar{i} \notin a \end{cases}$$

Under this setting, for example, three different vertical events in Figure 1 are mapped as follows:

$$(\{\bar{2}, \bar{6}, \bar{9}\}) \mapsto (0, 0, 4, 0, 0, 0, 3, 0, 0, 0, 0, 0)$$
$$(\{\bar{1}, \bar{4}, \bar{9}\}) \mapsto (0, 3, 0, 0, 5, 0, 0, 0, 0, 0, 0, 0)$$
$$(\{\bar{2}, \bar{4}, \bar{9}\}) \mapsto (0, 0, 2, 0, 5, 0, 0, 0, 0, 0, 0, 0)$$

In this mapping, points are close to each other if and only if their corresponding events involve similar intervals over the same pitches. Persistent homology analysis under this mapping usually shows non-trivial homology cycles in higher dimensions than mappings III and V, sometimes agreeing with mapping IV. In contrast with mapping III and IV, bars in barcodes corresponding to mappings V and VI are more scattered. Thus instead of forming blocks, they appear more individually.

The resulting persistence and barcodes diagrams of the first four measures of BWV 1048: I Allegro are shown in Figure 12 and Figure 13 respectively.

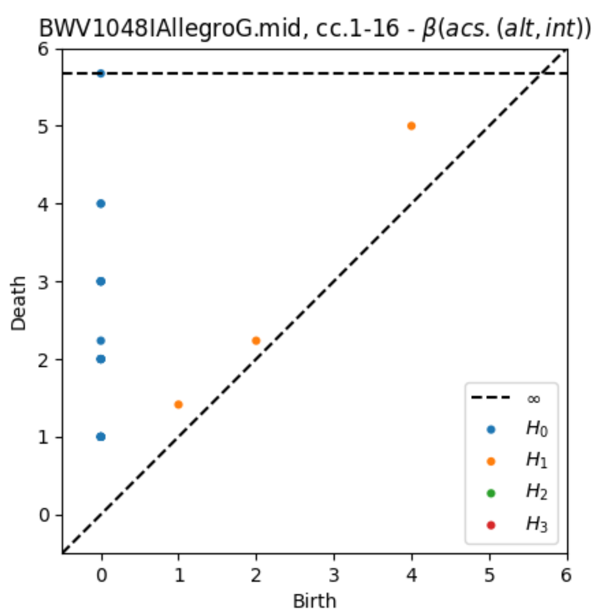

Figure 12: Persistence diagram of BWV 1048: I Allegro under Data Mapping VI.

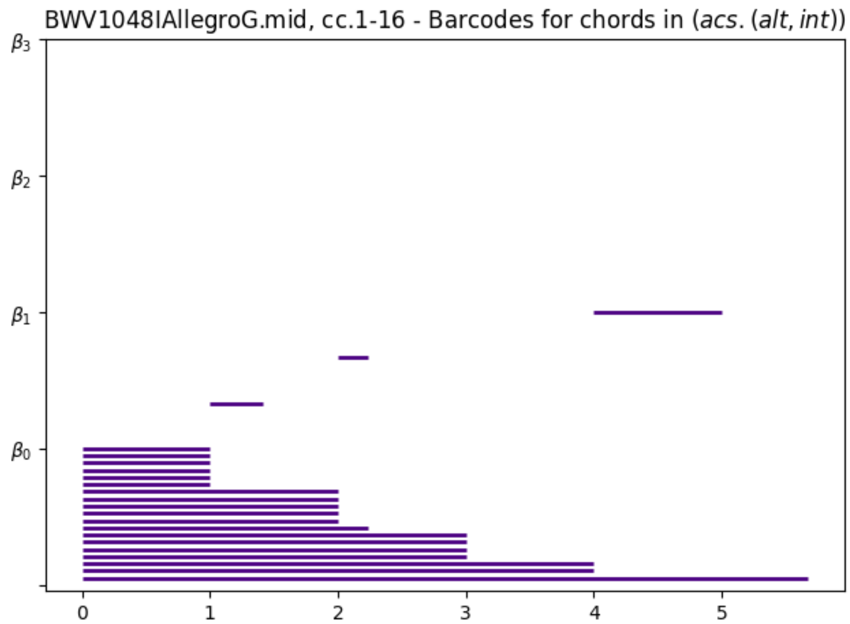

Figure 13: Barcode of BWV 1048: I Allegro under Data Mapping VI.

In summary, mappings IV, V and VI are different encodings of the normal form of chords. They are related to each other, since we may recover the pitch classes of a vertical event from any of these mappings. However, they yield diagrams with different levels of details, and capture different aspects of harmonic structures in a music fragment.

2.3 Two Harmonic Simplicial Complexes

In this section, we propose two ways to construct sequences of simplicial complexes from chord sequences, and compute their persistent homology without considering any metric among data points. The simplicial complexes here are formed by simplices or simplicial complexes representing individual chords, which are combined into a larger simplicial complex according to their order of appearance in the music fragment. We claim that for similar score samples, the associated complexes introduced here have similar Betti numbers. This will become more clear from the examples below.

2.3.1 Simplicial Complex of Cumulative Chords by Pitch

This representation captures the pitches of chords as vertices of simplices which are added together as events occur through the score.

Given an (n+1)-chord a with normal form vector $(\bar{x_0}, \bar{x_1}, \ldots, \bar{x_n})$, we difine its **associated n-simplex** as $s(a) = \{\bar{x_0}, \bar{x_1}, \bar{x_2}, \ldots, \bar{x_n}\}$. Given a music fragment $\mathcal{M} = \{e_0, e_1, \ldots, e_n\}$, we consider its sequence of chords $\mathcal{A}(\mathcal{M}) = \{a_0, a_1, \ldots, a_n\}$, which yields the sequence $s(a_0), s(a_1), \ldots, s(a_n)$ of associated simplices. We define the simplicial complex of cumulative chords by pitch in the

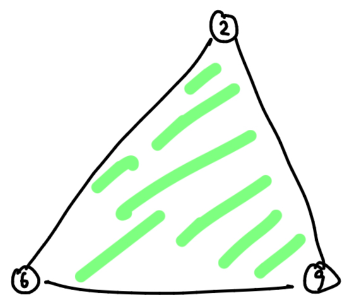

Figure 14: Building simplicial complexes $\mathcal{K}(0,0)$

interval of chords $[a_i, a_j]$, denoted by $\mathcal{K}(i,j)$, as the simplicial complex formed by the union of all associated simplices from $s(a_i)$ to $s(a_j)$, together with all their faces. However, this kind of setting is not sensitive to the order of appearance of chords. To fix this, we may consider the following sequence of simplicial complexes:

$$\mathcal{S}_a(\mathcal{M}) = \{\mathcal{K}(i,j)\}_{0\leq i\leq j\leq n}.$$

No matter the order of appearance of chords, we get the abstract simplex on the same set of pitch classes. However, the structure of the simplicial complex by pitches and intervals associated with a chord, which will be introduced later, will vary according to the order of pitches, and hence yielding a different topological encoding of the music fragment. This could be helpful when trying to preserve information of the actual intervals appearing in the score. In this case it may ve congruent to drop the octave-equivalence hypothesis and work directly with pitches rather than pitch classes. On the other hand, oriented simplices and simplicial complexes associated to vertical events could be considered in this framework, being interpreted as encoding the position of chords.

From now on, we will work with the main homological descirptors of ecah $\mathcal{K}(i,j)$, that is, their Betti numbers $\beta_k(\mathcal{K}(i,j))$ for $k = 0, 1, 2, \ldots$ and Euler characteristic $\chi(\mathcal{K}(i,j))$. Since the maximal dimension of simplices in $\mathcal{K}(i,j)$ is at most 11, we only need to consider Betti numbers up to dimension 11. We focus on the sequence of simplicial complexes $\mathcal{K}(0,j)$ for $j = 0, 1, \ldots, n$, which captures the cumulative harmonic content of the music fragment $\mathcal{M}$ as it progresses through time. In each step, we add a new simplex corresponding to the next chord in the sequence, along with all its faces.

Here is an example of the first four measures of BWV 1048: I Allegro. In the example score shown in Figure 1, let us show how we build $\mathcal{K}(0,0), \mathcal{K}(0,1), \mathcal{K}(0,2)$ from the following three chords:

$$(\{\bar{2}, \bar{6}, \bar{9}\}), (\{\bar{1}, \bar{4}, \bar{9}\}), (\{\bar{1}, \bar{2}, \bar{11}\}).$$

$\mathcal{K}(0,0)$ is just the simplex associated to the first chord, which is a 2-simplex with vertices $\bar{2}, \bar{6}, \bar{9}$, together with all its faces. Which is shown in the left of Figure 14.

To visualize it in $\mathbb{R}^2$, we have no choice but left the hall empty. But we have to keep in mind

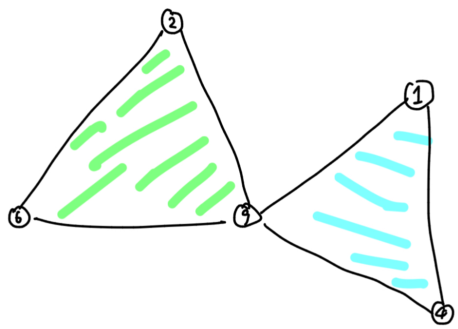

Figure 15: Building simplicial complexes $\mathcal{K}(0,1)$

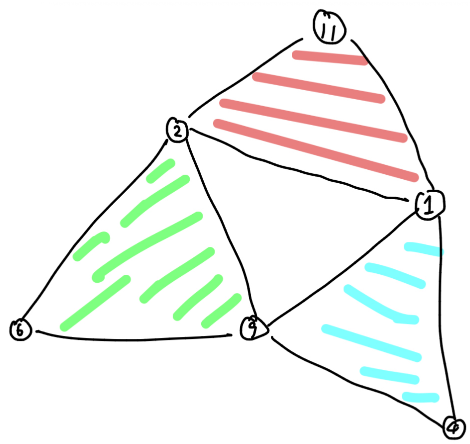

Figure 16: Building simplicial complexes $\mathcal{K}(0,2)$

that the triangle itself is actually contained in this complex, just as highlighted in green in the figure, no to mention higher dimensional faces.

Next, we build $\mathcal{K}(0,1)$ by adding the simplex associated to the second chord, which is also a 2-simplex with vertices $\bar{1}, \bar{4}, \bar{9}$, together with all its faces. Since vertex $\bar{9}$ is already present in $\mathcal{K}(0,0)$, we only need to add the edge $\{\bar{1}, \bar{9}\}$, the edge $\{\bar{4}, \bar{9}\}$, the edge $\{\bar{1}, \bar{4}\}$ and the face $\{\bar{1}, \bar{4}, \bar{9}\}$ to $\mathcal{K}(0,0)$, as shown in the left of Figure 15.

Finally, we build $\mathcal{K}(0,2)$ by adding the simplex associated to the third chord, which is a 2-simplex with vertices $\bar{1}, \bar{2}, \bar{1}\bar{1}$, together with all its faces. Since vertices $\bar{1}$ and $\bar{2}$ are already present in $\mathcal{K}(0,1)$, we only need to add the edge $\{\bar{1}, \bar{1}\bar{1}\}$, the edge $\{\bar{2}, \bar{1}\bar{1}\}$ and the face $\{\bar{1}, \bar{2}, \bar{1}\bar{1}\}$ to $\mathcal{K}(0,1)$, as shown in the left of Figure 16.

Clearly, after adding the third chord, we can see a 1-dimensional hole in the middle.

We show the barcode plot of Betti numbers for simplicial complexes $\mathcal{K}(0,j)$ associated with the first four measures of BWV 1048: I Allegro in Figure 17.

As a special case of the above, given an integer $r \in \{0, 1, \ldots, [n/2]\}$, and a sequence of chords $\mathcal{A}(\mathcal{M}) = \{a_0, a_1, \ldots, a_n\}$, we may consider the sequence of simplicial complexes $\mathcal{K}_r(i-r, i+r)$, correspinding to the interval of chords $[a_{i-r}, a_{i+r}]$. We call the complex $\mathcal{K}_r(i)$ the **simplicial**

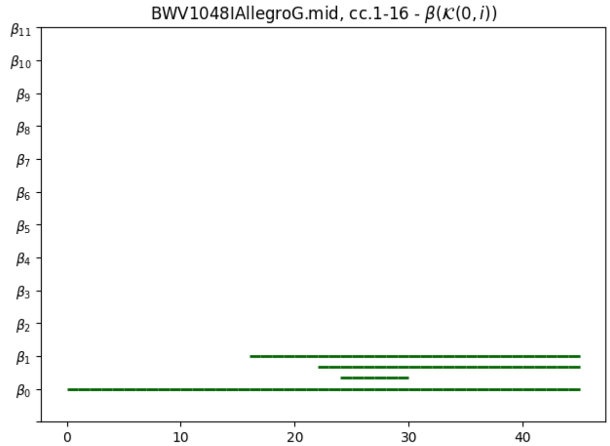

Figure 17: Barcode of Betti numbers of simplicial complexes by pitches of BWV 1048: I Allegro.

complex of events of radius r around chord a_i, who gives local information about harmonic sequences. Notice that the former one is not contained in the latter one, so they are not a filtration, but only a cover of it. So we may not strictly apply persistent homology techniques to them. Focusing on the subsequence of cunulative events of varying radii arounda fixed event a_i, we get the sequence $\{\mathcal{K}_r(i)\}_{r=0}^{[n/2]}$, which is a filtration. As an example, we show the Betti number of complexes of cumulative events of radius 4, $\mathcal{K}_4(i)$ in Figure 18.

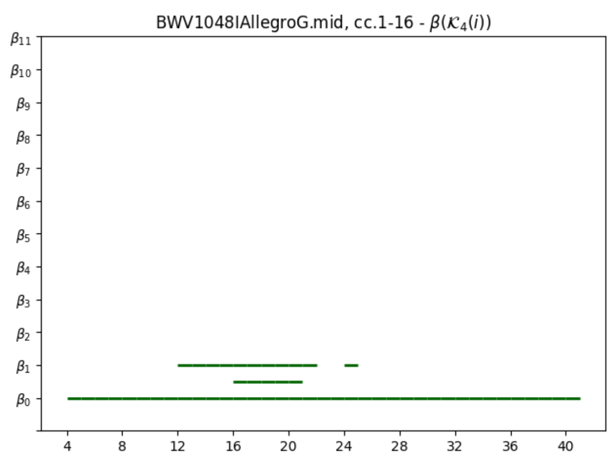

Figure 18: Barcode of Betti numbers of simplicial complexes of events of radius 4 of BWV 1048: I Allegro.

In the following works, we will focus on the results of calculating these homological descriptors for all possible radii. Note that simplicial complexes $\mathcal{K}(i,j)$ do not capture intervals in chords, which is a crucial harmonic feature. To fix this, we introduce another way to build simplicial complexes.

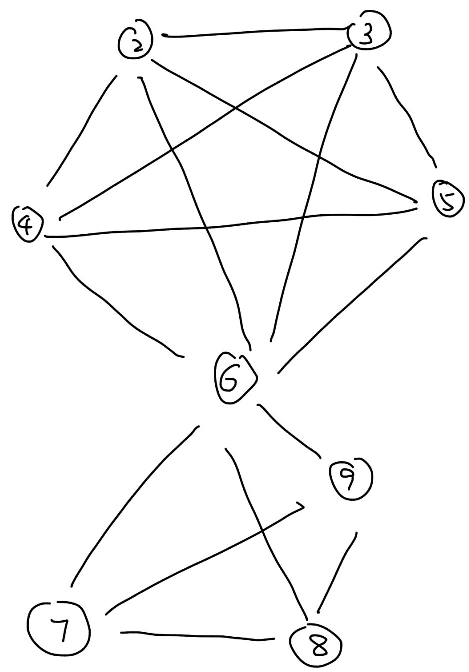

Figure 19: Simplicial complex associated to chord by pitches and intervals.

2.3.2 Simplicial Complex of Cumulative Chords by Pitch and Interval

We associate to a given chord a with normal form vector $(\bar{x_0}, \bar{x_1}, \dots, \bar{x_n})$, the simplicial complex whose simplices are $\sigma_i = \{\bar{x_i}, \bar{x_i}+1, \bar{x_i}+2, \dots, x_i \bar{+} 1\}$ together with all their faces, for $i = 0, 1, \dots, n-1$, where addition is taken mod 12. Given a sequence of chords $\mathcal{A}(\mathcal{M}) = \{a_0, a_1, \dots, a_n\}$, we denote the simplicial complex associated in this form to chord a_i by $math\tilde{c}alK(i,i)$, and proceed to define $\tilde{\mathcal{K}}(i,j)$ as teh previous section. Thus, from construction, we get a sequence of simplicial complexes representing the harmonic subsequence of $\mathcal{A}$, where we can run a homology analysis.

To visualize, take $(\{\bar{2}, \bar{6}, \bar{9}\})$ as an example again. Its associated simplicial complex $\tilde{\mathcal{K}}(i,i)$ is formed by three 1-simplices $\sigma_0 = \{\bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}\}$, $\sigma_1 = \{\bar{6}, \bar{7}, \bar{8}, \bar{9}\}$, together with all their faces, which is shown in Figure 19.

3 Results

To test our proposal as a way of describing and comparing musical data, we analyzed the persistence diagrams corresponding to the six data mappings described above, associated with four musical examples sharing some common stylistic features. Such fragments belong to the following baroque concertos: BWV 1048: I Allegro, BWV 1049: I Allegro, BWV 1050: I Allegro, Vivaldi Op.3 No.3: I Allegro, For this test, we only consider the first four measures of each fragment. We compare the harmonic data contained in these examples by using bottleneck distance calculated between their H_0-diagrams for all six data mappings. We only care about H_0-diagrams since no higer dimension diagrams were jointly generated for any pair of examples. We do not present all persistence and barcode diagrams here, but present the dendrograms that show the comparisons among H_0-diagrams of different examples under each data mapping.

In order to contrast our approach with the traditional tonal analysis, we briefly summarize the tonal relationships among these four examples using Roman Numeral Analysis, as the Figure 20 shows. When interpreting these progressions against our measurements, we must keep in mind that events may not always reflect this chords, since they may include harmonic ornaments as passing notes, retardations, etc. which are left out of the traditional analysis.

	m.1	m.2	m.3	m.4
BWV 1048	I	I	IV-ii-V$\frac{6}{5}$	I-vi-IV-ii
BWV 1049	I	V	I	V
BWV 1050	I	V	IV	V
RV 310	I	I-V^6-vi-V	IV-I^6-ii-I	V$\frac{6}{5}$-I-V^7-I

Figure 20: Roman Numeral Analysis of four baroque concertos.

We first consider the distance between H_0-persistence diagrams obtained from data mapping I, which incorporates pitches, onset times and durations. It is shown in Figure 21. By observation,

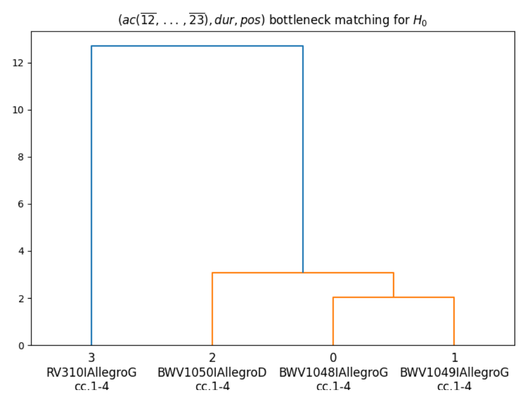

Figure 21: Data Mapping I.

we find that the first four measures of BWV 1048 and BWV 1050 are the closest in shape among all four samples. This is conpactible with the traditional tonal analysis, since the first movenments of both concertos are in the key of G major. Just as we discussed before, this data mapping is sensitive to transposition, so the closeness between these two examples is meaningful. Even though RV 310 is also in G major, it has the largest distance to other examples, which still coincides with the traditional analysis, since its harmonic rhythm and texture are quite different from the other three examples.

As for data mapping II and III, both of which uses interval vectors together, with and without onset times and durations respectively, the dendrogram of bottleneck distances between H_0-persistence diagrams is shown in Figure 22 and Figure 23 respectively. In both cases, we find that

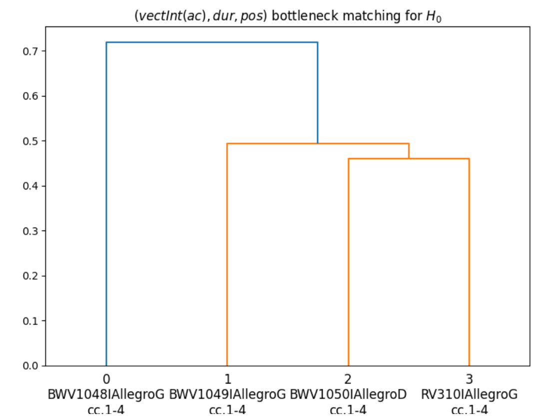

Figure 22: Data Mapping II.

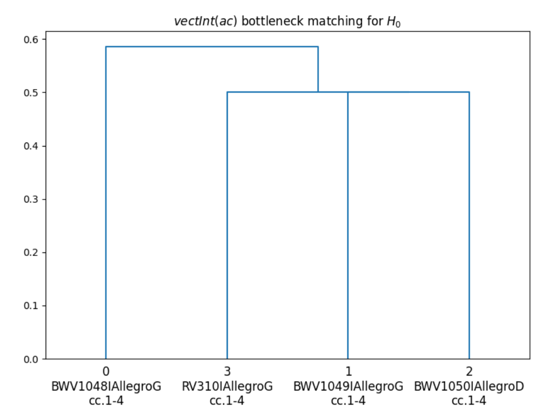

Figure 23: Data Mapping III.

RV 310 and BWV 1050 are the closest examples. After dropping onset times and durations, BWV 1049 become equal close to both examples. We may conclude that in the aspect of musical intervals, time information will affect the closeness among examples, especially for BWV 1049, which has a different position after removing time coordinates.

The result from data mapping IV, which only considers the presence of pitches in chords, is shown in Figure 24. Little information can be found in this case, since this mapping carries least

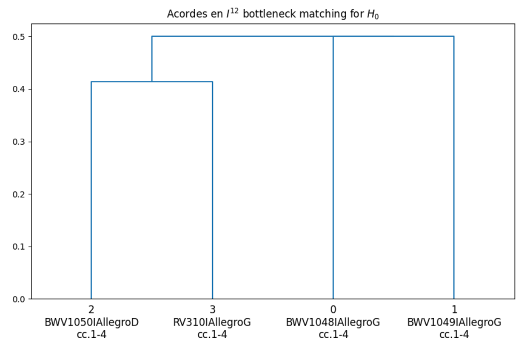

Figure 24: Data Mapping IV.

information about chords: only presence or absence of pitch classes. And hence once the fragments have been already similar enough, it of course fails.

The fifth mapping is the projection of the first mapping on its first 12 coordinates, which only considers pitches in chords. The dendrogram of bottleneck distances between H_0-persistence diagrams under this mapping is shown in Figure 25. Similar to data mapping I, we find that

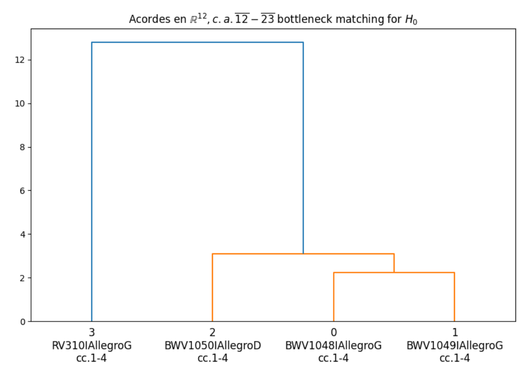

Figure 25: Data Mapping V.

BWV 1048 and BWV 1049 are the closest examples, which is consistent with the traditional tonal analysis. It is quite reasonable since these two ways care most about transposition, and by comparison, time information seems to have less effect on the closeness among examples. And removing time information even makes the result more significant.

Finally, we consider data mapping VI, which considers pitches and intervals between consecutive chords. The dendrogram of bottleneck distances between H_0-persistence diagrams under this mapping is shown in Figure 26. In this case, we find that the diagram are almost the same as data

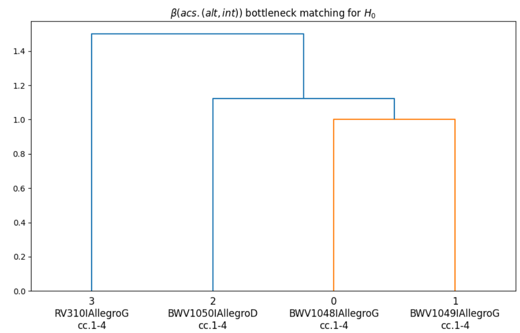

Figure 26: Data Mapping VI.

mapping V, but with low contrast. This can be explained as follows: in the fifth mapping, the results are significant, but when take musical intervals into consideration, with which we will get different results if separately considered, the overall effect is weakened.

4 Conclusion and Future Work

Overall, we followed the full procedure provided in [1] to analyze harmonic structures in four baroque concertos by using persistent homology. In that article, the authors compared Brandenburg Concerto No.1, 2, 3 with a mexican musician's work, and found that this way of analysis can capture stylistic differences among them. In our work, we choose to make a more consevative test by selecting four baroque concertos sharing similar stylistic features, and try to see whether our method can capture subtle differences among them. And finally we found that these methods not only coincides with traditional tonal analysis, but also provide us more information about how different aspects of harmonic structures affect the closeness among examples.

But these methods still lack quite a lot of musical features, such as melodic structures, voice leading, timbre, performance nuances, etc. In the future, we desire to find some ways to real performance audio data into this framework, and try to incorporate more musical features into the analysis.

A Persistent Homology

A.1 Simplicial Complexes

Definition 1. *A* simplicial complex K *is a collection of* simplices *such that if* $\sigma \in K$ *and* $\tau \subseteq \sigma$, *then* $\tau \in K$. *The dimension of a simplex* σ *is defined as* $\dim(\sigma) = |\sigma| - 1$, *where* $|\sigma|$ *is the number of vertices in* σ. *The dimension of the simplicial complex* K *is the maximum dimension of its simplices.*

For example, a 0-simplex is a vertex, a 1-simplex is an edge, a 2-simplex is a triangle, and a 3-simplex is a tetrahedron.

Definition 2. *A subcomplex L of a simplicial complex K is a subset of K that is itself a simplicial complex.*

Specifically, Vietoris-Rips complex is a kind of simplicial complex commonly used in persistent homology.

Definition 3. *Given a set of points X in a metric space and a distance parameter $\epsilon > 0$, the* Vietoris-Rips complex *$VR(X, \epsilon)$ is the simplicial complex where a simplex $\sigma = [v_0, v_1, \cdots, v_n]$ is included if and only if the pairwise distances between all points in σ are less than or equal to ϵ, in other words, $d(x_i, x_j) \leq \epsilon$ for all $x_i, x_j \in \sigma$.*

By observation, for $\forall r_1 < r_2$, we have $VR(X, r_1) \subseteq VR(X, r_2)$. This property allows us to construct a filtration of Vietoris-Rips complexes by varying the distance parameter ϵ.

Definition 4. *A map $f : K \to L$ between two simplicial complexes K and L is called a* simplicial map *if for every simplex $\sigma = [v_0, v_1, \cdots, v_n]$ in K, the image $f(\sigma) = [f(v_0), f(v_1), \cdots, f(v_n)]$ is a simplex in L.*

A.2 Homology

Homology is a mathematical concept used to study the topological features of a space, such as connected components, holes, and voids. It provides a way to quantify and classify these features using algebraic structures called homology groups.

Definition 5. *Let K be a simplicial complex, R be a ring, and n be a non-negative integer. Then we define $C_n(K; R)$ as the module over R generated by the set of n-simplices in K. The elements of $C_n(K; R)$ are called n-*chains.

Explicitly, a n-chain is a formal sum of n-simplices with coefficients in the ring R, which is a finit sum of the form $c = \sum_i r_i \sigma_i$, where $r_i \in R$ and σ_i are n-simplices in K.

To define homology groups, we need to introduce the boundary operator.

Definition 6. *The* boundary operator *$\partial_n : C_n(K; R) \to C_{n-1}(K; R)$ is a linear map defined on the basis elements (n-simplices) of $C_n(K; R)$ as follows:*

$$\partial_n([v_0, v_1, \cdots, v_n]) = \sum_{i=0}^{n} (-1)^i [v_0, v_1, \cdots, \hat{v_i}, \cdots, v_n],$$

where $\hat{v_i}$ indicates that the vertex v_i is omitted from the simplex.

The boundary operator captures the idea of the "boundary" of a simplex. For example, the boundary of a 1-simplex (edge) is its two endpoints, and the boundary of a 2-simplex (triangle) is its three edges. An important property of the boundary operator is that applying it twice yields zero, i.e., $\partial_{n-1} \circ \partial_n = 0$. This property allows us to define two important submodules of $C_n(K; R)$: the cycle group and the boundary group.

Definition 7. *The* cycle group $Z_n(K; R)$ *is defined as the kernel of the boundary operator* ∂_n*, i.e.,*

$$Z_n(K; R) = \ker(\partial_n) = \{c \in C_n(K; R) \mid \partial_n(c) = 0\}.$$

The elements of $Z_n(K; R)$ *are called* n-cycles*.*

Definition 8. *The* boundary group $B_n(K; R)$ *is defined as the image of the boundary operator* ∂_{n+1}*, i.e.,*

$$B_n(K; R) = \mathrm{im}(\partial_{n+1}) = \{\partial_{n+1}(c) \mid c \in C_{n+1}(K; R)\}.$$

The elements of $B_n(K; R)$ *are called* n-boundaries*.*

Since $\partial_{n-1} \circ \partial_n = 0$, we have $B_n(K; R) \subseteq Z_n(K; R)$. This inclusion allows us to define the homology groups.

Definition 9. *The* n-th homology group $H_n(K; R)$ *is defined as the quotient module*

$$H_n(K; R) = Z_n(K; R)/B_n(K; R).$$

The elements of $H_n(K; R)$ are equivalence classes of n-cycles modulo n-boundaries. The rank of the homology group $H_n(K; R)$, denoted as β_n, is called the *n-th Betti number*, which counts the number of n-dimensional holes in the simplicial complex K.

In practice, we often compute homology groups with coefficients in the field of two elements $\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$, to simplify calculations and ignore orientation issues.

Theorem 1. *Given a simplicial map* $f : K \to L$ *between two simplicial complexes* K *and* L*, the map* f *induces a homomorphism on the homology groups:*

$$f_* : H_n(K; R) \to H_n(L; R),$$

for each non-negative integer n*.*

A.3 Persistent Homology

Persistent homology is an extension of homology that studies the changes in topological features across multiple scales. It is particularly useful in data analysis, where one often deals with point clouds or other data sets that can be represented as filtrations of simplicial complexes. Given a filtration of simplicial complexes

$$K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_m,$$

we can compute the homology groups $H_n(K_i; R)$ for each complex K_i in the filtration. The inclusion maps $K_i \hookrightarrow K_j$ for $i \leq j$ induce homomorphisms on the homology groups:

$$\phi_{i,j} : H_n(K_i; R) \to H_n(K_j; R).$$

The persistent homology groups are defined as the images of these homomorphisms.

Definition 10. *The* n-th persistent homology group $H_n^{i,j}$ *is defined as*

$$H_n^{i,j} = \text{im}(\phi_{i,j}) \subseteq H_n(K_j; R).$$

The rank of the persistent homology group $H_n^{i,j}$, denoted as $\beta_n^{i,j}$, is called the *n-th persistent Betti number*, which counts the number of n-dimensional holes that persist from the complex K_i to the complex K_j.

In practice, we will consider filtations of Vietoris-Rips complexes

$$VR(X, \epsilon_0) \subseteq VR(X, \epsilon_1) \subseteq VR(X, \epsilon_2) \subseteq \cdots \subseteq VR(X, \epsilon_m),$$

where $0 = \epsilon_0 < \epsilon_1 < \epsilon_2 < \cdots < \epsilon_m = max\{d(x_i, x_j) : x_i, x_j \in X\}$.

Definition 11. *Given a homology class* $\alpha \in H_n(K_i; R)$*, its* birth *and* death *are defined as follows:*

- *The* birth *of* α *is the smallest index* b *such that* α *is in the image of the map* $H_n(K_b; R) \to H_n(K_i; R)$.
- *The* death *of* α *is the smallest index* $d > b$ *such that the image of* α *under the map* $H_n(K_i; R) \to H_n(K_d; R)$ *is zero.*

The *persistence* of α is defined as $d - b$. The multiset of all birth-death pairs (b, d) for homology classes in the filtration is called the *persistence diagram*.

Persistent homology provides a powerful tool for analyzing the topological features of data across multiple scales, allowing us to identify significant structures that persist over a range of parameters.

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