and assume $a>b>c>0$

3-3 #20 Let us follow the hint given in the book.

Let $F(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$. Then $\vec{\nabla}F = \left(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2}\right)$ is a normal field over the ellipsoid, or we can take

$fN = \left(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}\right)$, including f so we do not have to normalize and mess up with square roots. We compute that $\frac{d(fN)}{dt} = \frac{df}{dt}\cdot N + f\cdot\frac{dN}{dt}$ and note $\left\langle \frac{df}{dt}\cdot N \wedge -, N\right\rangle \equiv 0$. since in the mixed product the first and third terms are linearly dependent. Therefore

at an umbilical point, $\left\langle \frac{d(fN)}{dt} \wedge \frac{d\alpha}{dt}, N\right\rangle =$

$\left\langle f\cdot\frac{dN}{dt} \wedge \frac{d\alpha}{dt}, N\right\rangle = \left\langle f\cdot k\frac{d\alpha}{dt} \wedge \frac{d\alpha}{dt}, N\right\rangle = 0$ where

k is the single eigenvalue (with multiplicity 2) and any tangent vector $\frac{d\alpha}{dt}$ is an eigenvector.

Now, we plug in $fN = \left(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}\right)$ and $\alpha = (x,y,z)$ so that

$$\left\langle \frac{d(fN)}{dt} \wedge \frac{d\alpha}{dt}, fN\right\rangle = \begin{vmatrix} \frac{x'}{a^2} & \frac{y'}{b^2} & \frac{z'}{c^2} \\ x' & y' & z' \\ \frac{x}{a^2} & \frac{y}{b^2} & \frac{z}{c^2} \end{vmatrix} = 0 \qquad (*)$$

Let us first assume that $z \neq 0$. Then $(*)$ becomes

$$0 = \begin{vmatrix} \dfrac{x'}{a^2} & \dfrac{y'}{b^2} & \dfrac{z'}{c^2} \\ x' & y' & z' \\ \dfrac{xc^2}{za^2} & \dfrac{yc^2}{zb^2} & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{x'}{a^2} - \dfrac{xz'}{za^2} & \dfrac{y'}{b^2} - \dfrac{yz'}{zb^2} & 0 \\ x' - \dfrac{xz'c^2}{za^2} & y' - \dfrac{yz'c^2}{zb^2} & 0 \\ \dfrac{xc^2}{za^2} & \dfrac{yc^2}{zb^2} & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{x'}{a^2} - \dfrac{xz'}{za^2} & \dfrac{y'}{b^2} - \dfrac{yz'}{zb^2} \\ x' - \dfrac{xz'c^2}{za^2} & y' - \dfrac{yz'c^2}{zb^2} \end{vmatrix} \qquad (**)$$

Note that, since $\alpha'(t)$ is arbitrary, we may take $y' = 0$ so that

$$0 = \begin{vmatrix} \dfrac{x'}{a^2} - \dfrac{xz'}{za^2} & -\dfrac{yz'}{zb^2} \\ x' - \dfrac{xz'c^2}{za^2} & -\dfrac{yz'c^2}{zb^2} \end{vmatrix}$$

$$= c^2 \begin{vmatrix} \dfrac{x'}{a^2} - \dfrac{xz'}{za^2} & -\dfrac{yz'}{zb^2} \\ \dfrac{x'}{c^2} - \dfrac{xz'}{za^2} & -\dfrac{yz'}{zb^2} \end{vmatrix}$$

$$= -\frac{c^2 y z'}{zb^2} \begin{vmatrix} \dfrac{x'}{a^2} - \dfrac{xz'}{za^2} & 1 \\ \dfrac{x'}{c^2} - \dfrac{xz'}{za^2} & 1 \end{vmatrix}$$

$$= -\frac{c^2 y z'}{zb^2} \, x' \left(\frac{1}{a^2} - \frac{1}{c^2} \right)$$

and conclude that $y = 0$. We then have

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$$

and so

$$\frac{2xx'}{a^2} + \frac{2zz'}{c^2} = 0$$

$$\frac{xx'}{a^2} = -\frac{zz'}{c^2}$$

$$\frac{x'}{z'} = -\frac{za^2}{xc^2} \qquad (***)$$

~~Plugging this into (**)~~, We then obtain from (**)

$$0 = \begin{vmatrix} \frac{x'}{a^2} - \frac{xz'}{za^2} & \frac{y'}{b^2} \\ x' - \frac{xz'c^2}{za^2} & y' \end{vmatrix} = \frac{1}{b^2} \begin{vmatrix} \frac{x'b^2}{a^2} - \frac{xz'b^2}{za^2} & y' \\ x' - \frac{xz'c^2}{za^2} & y' \end{vmatrix}$$

which forces

$$\frac{x'b^2}{a^2} - \frac{xz'b^2}{za^2} = x' - \frac{xz'c^2}{za^2}$$

$$\Rightarrow x'\left(\frac{b^2}{a^2} - 1\right) = \frac{xz'}{z}\,\frac{b^2-c^2}{a^2}$$

$$\Rightarrow \frac{x'}{z'}(b^2 - a^2) = \frac{x}{z}(b^2-c^2)$$

$$\overset{(***)}{\Rightarrow} -\frac{za^2}{xc^2}(b^2-a^2) = \frac{x}{z}(b^2-c^2)$$

$$\Rightarrow z^2a^2(a^2-b^2) = x^2c^2(b^2-c^2)$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{x^2(b^2-c^2)}{a^2(a^2-b^2)} = 1$$

$$\Rightarrow \quad x^2 \frac{a^2-c^2}{a^2-b^2} = a^2$$

$$\Rightarrow \quad x = \pm\sqrt{\frac{a^2(a^2-b^2)}{a^2-c^2}}$$

$$\Rightarrow \quad z = \pm\sqrt{\frac{c^2(b^2-c^2)}{a^2-c^2}}$$

Thus we get four umbilical points $\left(\pm a\sqrt{\frac{a^2-b^2}{a^2-c^2}}, 0, \pm c\sqrt{\frac{b^2-c^2}{a^2-c^2}}\right)$.

(If we instead set $x'=0$ on page 2, then $x=0$, which leads similarly to $-z^2 b^2(a^2-b^2) = y^2c^2(a^2-c^2)$, contradicting $a>b>c>0$. ~~Similarly ... $z=0$ will not yield more umbilical points.~~)

Finally, if $z=0$ (cf. bottom of page 1), (*) becomes

$$0 = \begin{vmatrix} \frac{x'}{a^2} & \frac{y'}{b^2} & \frac{z'}{c^2} \\ x' & y' & z' \\ \frac{x}{a^2} & \frac{y}{b^2} & 0 \end{vmatrix}$$

Further taking $x'=0$, we obtain

$$0 = \begin{vmatrix} 0 & \frac{y'}{b^2} & \frac{z'}{c^2} \\ 0 & y' & z' \\ \frac{x}{a^2} & \frac{y}{b^2} & 0 \end{vmatrix} = \frac{x}{a^2}\, y'z'\left(\frac{1}{b^2}-\frac{1}{c^2}\right)$$

which forces $x=0$. However, $(0,\pm b, 0)$ are clearly not umbilical points.

To conclude, there are exactly four umbilical points located on the ellipse $\frac{x^2}{a^2}+\frac{z^2}{c^2}=1$, $y=0$.