MA327 Midterm Exam

Name: ______________________

Instructions: Calculators, course notes and textbooks are **NOT** allowed on the worksheet. All numerical answers **MUST** be exact; e.g., you should write π instead of 3.14..., $\sqrt{2}$ instead of 1.414..., and $\frac{1}{3}$ instead of 0.3333... Explain your reasoning using complete sentences and correct grammar, spelling, and punctuation.
Show ALL of your work!

Question 1. Show that the curve

$$\alpha(t) = (t, \sin t, -\cos t)$$

has constant speed. Then find a re-parametrization of this curve by arc length.

We compute $\alpha'(t) = (1, \cos t, \sin t)$ and hence

$|\alpha'(t)| = \sqrt{1^2 + \cos^2 t + \sin^2 t} = \sqrt{2}$ is constant.

Since the arc length $s(t) = \int_0^t |\alpha'(t)|\,dt = \sqrt{2}\,t$,

we have $t = \frac{s}{\sqrt{2}}$, so a re-parametrization

by s is given by $\alpha(s) = \left(\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, -\cos\frac{s}{\sqrt{2}}\right)$.

Question 2. Write down the Frenet formulas for the derivatives of the tangent, normal, and binormal unit vectors of a curve parametrized by arc length. Be careful with notation. Explain all the symbols involved as well as the precise assumptions on the curve.

Let $t(s)$, $n(s)$, and $b(s)$ be the unit tangent, normal, and binormal vectors of the curve α with arc length parameter s. Suppose that $\alpha'(s) \neq 0$ and $\alpha''(s) \neq 0$ for all s.

Then

$$\begin{cases} t'(s) = & & k(s)n(s) & \\ n'(s) = & -k(s)t(s) & & -\tau(s)b(s) \\ b'(s) = & & \tau(s)n(s) & \end{cases}$$

where $k(s)$ is the curvature and $\tau(s)$ is the torsion.

Question 3. Consider the coordinate chart

$$\mathbf{x}(u, v) = (u^3, u^2 + v^2, v^3)$$

for $u, v > 0$. Find the coefficients E, F, and G of the first fundamental form in these coordinates.

We compute

$$\vec{X}_u = (3u^2, 2u, 0)$$

$$\vec{X}_v = (0, 2v, 3v^2)$$

Therefore

$$E = \langle \vec{X}_u, \vec{X}_u \rangle = 9u^4 + 4u^2$$

$$F = \langle \vec{X}_u, \vec{X}_v \rangle = 4uv$$

$$G = \langle \vec{X}_v, \vec{X}_v \rangle = 4v^2 + 9u^4$$

Question 4. Suppose we have a coordinate chart $\mathbf{x}$ on the open set

$$\{(u,v) \in \mathbb{R}^2 \mid u > 0,\ 0 < v < 2\pi\}$$

such that the coefficients of the first fundamental form are:

- $E = e^{-u}$,
- $F = 0$,
- $G = e^{-u}$.

(a) Compute the length of the image under $\mathbf{x}$ of the curve $\alpha(t) = (2,t)$ between $t = 0$ and $t = 1$.

The length is given by

$$\int_0^1 \left| \frac{d}{dt} \vec{x}(\alpha(t)) \right| dt = \int_0^1 \left| \frac{d}{dt} \vec{x}(2,t) \right| dt$$

$$= \int_0^1 \left| \vec{x}_u(2,t) \cdot 0 + \vec{x}_v(2,t) \cdot 1 \right| dt$$

$$= \int_0^1 \sqrt{G(2,t)}\, dt$$

$$= \int_0^1 \sqrt{e^{-2}}\, dt$$

$$= e^{-1}$$

(b) Find the area of the image of the entire region.

The area can be computed as

$$\int_0^\infty \int_0^{2\pi} |\vec{x}_u \wedge \vec{x}_v|\, dv\, du = \int_0^\infty \int_0^{2\pi} \sqrt{EG - F^2}\, dv\, du$$

$$= \int_0^\infty \int_0^{2\pi} e^{-u}\, dv\, du$$

$$= 2\pi$$

Question 5. Find all possible fields of unit normal vectors on the surface given by $z^2 - x^2 - y^2 = 1$.

Let $f(x,y,z) := x^2 + y^2 - z^2 + 1$. Then $\vec{\nabla} f(x,y,z) = (2x, 2y, -2z)$ is a normal vector to the surface. Thus the unit normal vector fields are

$$N(x,y,z) = \frac{1}{\sqrt{4x^2+4y^2+4z^2}}(2x,2y,-2z)$$
$$= \frac{1}{\sqrt{x^2+y^2+z^2}}(x,y,-z)$$

and $$\bar{N}(x,y,z) = -\frac{1}{\sqrt{x^2+y^2+z^2}}(x,y,-z).$$

Question 6. Given a regular surface S with unit normal vector field N and a point $p \in S$, state carefully the mathematical definition of the second fundamental form II_p of S at p.

Define the second fundamental form $II_p(v) := \langle -dN_p(v), v\rangle$ for any $v \in T_p(S)$, where $dN_p : T_p(S) \longrightarrow T_{N(p)}(S^2) = T_p(S)$ is the differential of the Gauss map $N : S \to S^2$ at p.

Question 7. Suppose that N is a field of unit normal vectors on a surface S, and $f : S \to \mathbb{R}$ a smooth function with $f(p) > 0$ for all p in the surface. Define $M(p) = f(p)N(p)$. This is a field of normal vectors which are not necessarily unit vectors.

(a) Show that $\langle dM_p(v), v\rangle = -f(p)II_p(v)$ for each $v \in T_p(S)$.

Proof We have

$$\begin{aligned}\langle dM_p(v), v\rangle &= \langle df_p(v)\, N(p) + f(p)\, dN_p(v), v\rangle \\ &= df_p(v)\langle N(p), v\rangle + f(p)\langle dN_p(v), v\rangle \\ &= df_p(v)\cdot 0 + f(p)(-II_p(v)) \\ &= -f(p)\, II_p(v). \qquad \square\end{aligned}$$

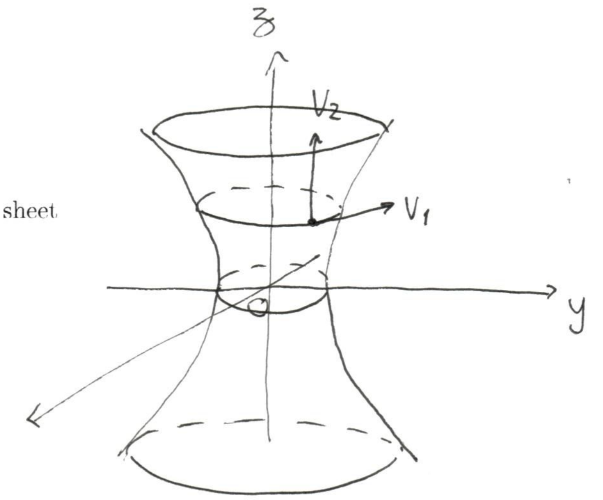
(b) Show that the point p is hyperbolic if and only if $\langle dM_p(v), v\rangle$ takes on both positive and negative values for different choices of v.

Proof Recall that p is hyperbolic if $II_p(v)$ takes on both positive and negative values for different choices of vector v (when the linear map dN_p has eigenvalues of distinct signs). Since $f(p) > 0$, the conclusion then follows from the identity proved in (a). $\square$

Question 8. Show that all points on the hyperboloid of one sheet

$$x^2 + y^2 - z^2 = 1$$

are hyperbolic.



Proof Let $f(x,y,z) := x^2+y^2-z^2$. Then we can take $M(p)$ in Question 7 to be $M(x,y,z) = \vec{\nabla} f(x,y,z) = (2x, 2y, -2z)$. Thus $dM_p : T_p(S) \to \mathbb{R}^3$ sends $v = \alpha'(0) = (x'(0), y'(0), z'(0))$ to $(M \circ \alpha)'(0) = (2x'(0), 2y'(0), -2z'(0))$.

Therefore, given any $v = (a, b, c) \in T_p(S)$, we have $\langle dM_p(v), v\rangle = 2a^2 + 2b^2 - 2c^2$.

If $v_1 = (y, -x, 0) \in T_p(S)$, then

$$\langle dM_p(v_1), v_1 \rangle = 2y^2 + 2x^2 = 2 + 2z^2 > 0.$$

When $z \neq 0$, if $v_2 = \left(x, y, \frac{x^2+y^2}{z}\right) \in T_p(S)$,

Then $\langle dM_p(v_2), v_2\rangle = -\frac{2(1+z^2)}{z^2} < 0$.

When $z = 0$, if $v_2 = (0, 0, 1) \in T_p(S)$,

then $\langle dM_p(v_2), v_2 \rangle = -2 < 0$.

By Question 7(b), each point (x, y, z) is hyperbolic. $\square$

Question 9. Show that the sphere of radius $a > 0$ centered at the origin has constant Gaussian curvature $1/a^2$ and mean curvature $1/a$.

Proof The Gauss map N sends (x, y, z) to the vector $\frac{1}{a}(x, y, z)$ (or $-\frac{1}{a}(x, y, z)$, depending on the orientation of the sphere).

Thus its differential is the map of multiplication by $\frac{1}{a}$ (or $-\frac{1}{a}$). This linear map has a single eigenvalue $\frac{1}{a}$ (or $-\frac{1}{a}$). Therefore the principal curvatures $k_1 = k_2 = \frac{1}{a}$ (or $-\frac{1}{a}$), and so the Gaussian curvature $k_1 k_2 = \frac{1}{a^2}$ and the mean curvature $\frac{1}{2}(k_1 + k_2) = \frac{1}{R}$ (or $-\frac{1}{R}$) for any point on the sphere. $\square$

Question 10. Suppose you have a curve $\alpha(t)$ in a surface S with normal vector field N on the surface.

(a) Show that if $\alpha''(t)$ is always parallel to $N_{\alpha(t)}$, then the length of $\alpha'(t)$ is constant.

Proof Since $\alpha''(t)$ and $N_{\alpha(t)}$ are parallel and $\alpha'(t)$ and $N_{\alpha(t)}$ are onthogonal, we know $\langle \alpha''(t), \alpha'(t) \rangle = 0$. Thus

$$\frac{d}{dt} |\alpha'(t)|^2 = 2\langle \alpha''(t), \alpha'(t) \rangle = 0$$

and so $|\alpha'(t)|$ is constant. $\square$

(b) Is the converse to the statement in part (a) true? Give either a proof or a counterexample.

No. Let S be a plane and α be any regular curve parametrized by arc length s. Then $|\alpha'(s)| = 1$ is constant but $\alpha''(t)$ is always onthogonal to $N_{\alpha(t)}$.