#1 #2 #3 #4 #5 #6 Total
8.4/10 13.2/20 8.1/15 12.3/20 14.6/25 5.1/10 61.6/100
Median 61.5

90-100: × × × ×
80-89: × × × × × ×
70-79: × × × × × × × ×
60-69: × × × | × × × × × ×
50-59: × × × × × × ×
40-49: × × ×
<40: × × × × × × × × × ×

MA323 Topology Midterm Exam

Suggested Grading Scheme and Solutions

November 6, 2026

Total: 100 points. This is a suggestion, not a straitjacket: accept equivalent correct proofs, and award partial credit for a correct key step even if the write-up is inefficient. For counterexample items, a named standard example is acceptable only when the requested justification is actually supplied.

Question 1 (10 points)

Problem. *Let X be a topological space and $x \in X$.*

1. *(5 points) Define what it means for a collection $\mathscr{B}_x$ of open subsets of X to be a basis of open neighborhoods at x.*
2. *(5 points) Suppose that X is first countable. Prove that there is a* decreasing *countable basis $\{V_n\}_{n=1}^{\infty}$ of open neighborhoods at x, i.e., $V_1 \supset V_2 \supset \cdots$.*

Model solution. (a) A collection $\mathscr{B}_x$ is a basis of open neighborhoods at x if every $B \in \mathscr{B}_x$ is open and contains x, and if for every open set U with $x \in U$ there is some $B \in \mathscr{B}_x$ such that $x \in B \subset U$. Equivalently: the basic open neighborhoods are cofinal among all neighborhoods of x.

(b) Let $\{U_n\}_{n\in\mathbb{N}}$ be a countable basis of open neighborhoods at x. Put $V_n = U_1 \cap \cdots \cap U_n$. Each V_n is open, contains x, and $V_1 \supset V_2 \supset \cdots$. If W is open and $x \in W$, choose n with $U_n \subset W$; then $V_n \subset U_n \subset W$. Thus $\{V_n\}$ is a countable decreasing basis of open neighborhoods at x.

Suggested rubric. (a) 5 points: 2 for requiring basic elements to be open neighborhoods of x, 3 for the cofinality/containment condition. (b) 5 points: 2 for finite intersections, 1 for monotonicity, 2 for checking the basis property.

Accept / watch for. In (b), the proof must not merely intersect with an arbitrary open W; the decreasing family has to be built once, before W is given.

Question 2 (20 points)

Problem. *Let X be a topological space, $A \subset X$, and $x \in X$.*

1. *(5 points) Prove that if there is a sequence $\{a_n\}_{n=1}^{\infty}$ in A converging to x, then $x \in \overline{A}$.*
2. *(5 points) Prove that if X is metrizable and $x \in \overline{A}$, then there is a sequence $\{a_n\}_{n=1}^{\infty}$ in A converging to x.*
3. *(10 points) Give an example, with justification, of a space X, $A \subset X$, and $x \in \overline{A}$ for which no sequence in A converges to x. Conclude that X is not metrizable.*

Model solution. (a) Suppose $a_n \in A$ and $a_n \to x$. If U is any neighborhood of x, then eventually $a_n \in U$; in particular $U \cap A \neq \varnothing$. Hence $x \in \overline{A}$.

(b) Assume X is metrizable and $x \in \overline{A}$. For each n, the ball $B(x, 1/n)$ is a neighborhood of x, so $A \cap B(x, 1/n) \neq \varnothing$. Choose a_n in this intersection. Then $d(a_n, x) < 1/n$, so $a_n \to x$.

(c) Let X be uncountable with the cocountable topology, fix $x \in X$, and let $A = X \setminus \{x\}$. Every nonempty open set is cocountable, so every neighborhood of x meets A; thus $x \in \overline{A}$. But if $\{a_n\}_{n=1}^{\infty}$ is any sequence in A, then $X \setminus \{a_1, a_2, \dots\}$ is an open neighborhood of x containing no term of the sequence, so $a_n \not\to x$. By the contrapositive of part (b), such an X is not metrizable.

Suggested rubric. (a) 5 points for the neighborhood-eventually argument. (b) 5 points: 2 for nonemptiness from closure, 3 for the $1/n$ convergence. (c) 10 points: 4 for a valid example with $x \in \overline{A}$, 4 for proving no sequence in A converges to x, 2 for explicitly concluding non-metrizability from part (b).

Accept / watch for. The converse direction needs a choice of points a_n; merely saying "x is a limit point" is not enough. In (c), the finite-complement topology on an uncountable set is not a valid substitute: there nontrivial sequences do converge. Also, "X is not metrizable" should be justified by part (b), not just asserted.

Additional accepted example for Question 2(c): the minimal uncountable well-ordered set.

Let S_Ω be the minimal uncountable well-ordered set (Munkres, §10), and equip $X = S_\Omega \cup \{\Omega\}$, where Ω is adjoined as a greatest element, with the order topology. Set $A = S_\Omega$ and $x = \Omega$. Every neighborhood of Ω contains $(\alpha, \Omega]$ for some $\alpha \in S_\Omega$, and this interval meets A; hence $\Omega \in \mathrm{cl}_X(A)$. For any sequence (a_n) in A, the set of its terms is countable and therefore has an upper bound $\beta \in S_\Omega$. Indeed, an unbounded countable set would express S_Ω as a countable union of countable initial segments. Thus $(\beta, \Omega]$ is an open neighborhood of Ω containing no a_n, so a_n does not converge to Ω. By part (b), X is not metrizable.

Accept / watch for. Apply the same 4 + 4 + 2 point breakdown as above. A named example alone is insufficient: require justification of both the closure claim and the countable-upper-bound obstruction.

Question 3 (15 points)

Problem. *Let X be a topological space.*

1. *(10 points) Let $\Delta_2 \colon X \to X \times X$ be its diagonal map, so that $\Delta_2(x) = (x, x)$. Prove that X is Hausdorff if and only if Δ_2 is a closed embedding.*
2. *(5 points) Does the same criterion hold for $\Delta_3 \colon X \to X \times X \times X$ with $\Delta_3(x) = (x, x, x)$? Prove it or give a counterexample.*

Model solution. (a) Let $\Delta_2(x) = (x, x)$. The map Δ_2 is always continuous, since its two coordinate maps are the identity; it is injective, and the inverse $\Delta_2(X) \to X$ is the restriction of the first projection, hence continuous. So Δ_2 is always an embedding. The only issue is whether $\Delta_2(X)$ is closed.

If X is Hausdorff and $(x, y) \notin \Delta_2(X)$, then $x \neq y$. Choose disjoint open $U \ni x$ and $V \ni y$. Then $U \times V$ is a neighborhood of (x, y) disjoint from $\Delta_2(X)$. Thus $\Delta_2(X)$ is closed.

Conversely, if $\Delta_2(X)$ is closed and $x \neq y$, then (x, y) lies in the open complement of $\Delta_2(X)$. Choose a basic product open set $U \times V$ with $(x, y) \in U \times V \subset (X \times X) \setminus \Delta_2(X)$. Then $U \ni x$, $V \ni y$, and $U \cap V = \varnothing$; otherwise $z \in U \cap V$ would give $(z, z) \in (U \times V) \cap \Delta_2(X)$. Hence X is Hausdorff.

(b) Yes, the same criterion holds for $\Delta_3(x) = (x, x, x)$. Again Δ_3 is automatically an embedding. If X is Hausdorff and $(x_1, x_2, x_3) \notin \Delta_3(X)$, then some two coordinates differ, say $x_i \neq x_j$. Separate them by disjoint open sets and put those sets in coordinates i and j (put X in the remaining coordinate); the resulting product neighborhood misses $\Delta_3(X)$. Conversely, if $\Delta_3(X)$ is closed and $x \neq y$, then $(x, x, y) \notin \Delta_3(X)$. A basic open $U_1 \times U_2 \times U_3$ about (x, x, y) missing $\Delta_3(X)$ gives disjoint neighborhoods $U_1 \cap U_2$ of x and U_3 of y.

Suggested rubric. (a) 10 points: 3 for noting the diagonal is automatically an embedding, 3 for Hausdorff $\Rightarrow$ closed diagonal, 4 for closed diagonal $\Rightarrow$ Hausdorff. (b) 5 points: 2 for the correct yes answer, 3 for adapting the argument by choosing a pair of differing coordinates.

Accept / watch for. In (b), a triple is off the diagonal when not all coordinates are equal; students should isolate one unequal pair rather than trying to separate all three points at once. Merely saying "same proof" earns little without the coordinate observation.

Question 4 (20 points)

Problem. *Let X be compact.*

1. *(5 points) T_2 implies T_3: Suppose X is Hausdorff. If $C \subset X$ is closed and $x \notin C$, construct disjoint open sets U and V with $x \in U$ and $C \subset V$.*
2. *(5 points) T_3 implies T_4: Suppose X is T_3. Show that X is in fact T_4 by separating each pair of disjoint closed subsets of X with disjoint open sets.*
3. *(10 points) Without compactness, the chain of implications breaks down. Give an example, with justification, showing either $T_2 \not\Rightarrow T_3$ or $T_3 \not\Rightarrow T_4$.*

Model solution. (a) Suppose X is T_2. Let $C \subset X$ be closed and $x \notin C$. Since X is compact, C is compact. For each $y \in C$, the T_2 axiom gives disjoint open sets $U_y \ni x$ and $V_y \ni y$. The sets V_y cover C, so finitely many $V_{y_1}, \dots, V_{y_n}$ already cover C. Put $U = U_{y_1} \cap \cdots \cap U_{y_n}$ and $V = V_{y_1} \cup \cdots \cup V_{y_n}$. Then $x \in U$, $C \subset V$, and $U \cap V = \varnothing$.

(b) Suppose X is T_3, and let $A, B \subset X$ be disjoint and closed. Since A is closed in the compact space X, it is compact. For each $a \in A$, the T_3 axiom separates the point a from the closed set B: there are disjoint open sets $U_a \ni a$ and $V_a \supset B$. Finitely many $U_{a_1}, \dots, U_{a_m}$ cover A. Put $U = U_{a_1} \cup \cdots \cup U_{a_m}$ and $V = V_{a_1} \cap \cdots \cap V_{a_m}$. Then $A \subset U$, $B \subset V$, and $U \cap V = \varnothing$.

(c) One failure outside compactness is $\mathbb{E}_K$: it is T_2 because it is finer than the usual topology, and $K = \{1/n : n \in \mathbb{Z}_+\}$ is closed, but 0 cannot be separated from K by disjoint open sets, so $\mathbb{E}_K$ is not T_3. Thus $T_2 \not\Rightarrow T_3$ without compactness. A Moore-plane type example can instead be used for $T_3 \not\Rightarrow T_4$ if the class has covered it; require only one correct failure.

Suggested rubric. (a) 5 points: 1 for compactness of C, 3 for the finite-subcover construction, 1 for disjointness check. (b) 5 points: 1 for compactness of A, 2 for using the T_3 axiom to separate each point of A from B, 2 for the finite union/intersection check. (c) 10 points: 4 for a correct example, 6 for the justification identifying the failed separation axiom.

Accept / watch for. Do not accept "use compactness" without the finite subcover step in (a) or (b). In (b), the pointwise separations should come from the T_3 axiom, not merely from T_2. In (c), a bare name is not enough; the answer should say which implication fails and why. There is no contradiction with (a)–(b): those are under compactness.

Additional accepted examples for Question 4(c).

Two other examples are: (i) Chengye You, Ji Chu Tuo Pu Xue Jiang Yi, Ch. 2, §1, Exercise 18. Its topology $\tau = \{U \setminus A : U$ is Euclidean-open, A is a subset of the irrationals$\}$ is T_2 but not T_3. (ii) The Sorgenfrey plane (the product of two lower-limit lines) is T_3 but not T_4. The relevant separation properties must be justified.

Question 5 (25 points)

Problem. *Recall that a bijection $f\colon X \to Y$ is a homeomorphism when both f and f^{-1} are continuous.*

1. *(5 points) Let $f\colon X \to Y$ be a continuous bijection with X compact and Y Hausdorff. Prove that f is a homeomorphism.*
2. *(5 points) Show that the compactness hypothesis in part (a) cannot be omitted: prove that*

$$f\colon [0, 2\pi) \to S^1, \qquad t \mapsto (\cos t, \sin t),$$

   is a continuous bijection but not a homeomorphism.
3. *(5 points) Show that the Hausdorff hypothesis in part (a) cannot be omitted: construct a continuous bijection from a compact space onto a space that is not Hausdorff, which is not a homeomorphism.*
4. *(10 points) Let $\mathscr{T}_1 \subset \mathscr{T}_2$ be two topologies on the same set X, with $(X, \mathscr{T}_2)$ compact and $(X, \mathscr{T}_1)$ Hausdorff. Prove that $\mathscr{T}_1 = \mathscr{T}_2$. Conclude that if $(X, \mathscr{T})$ is compact Hausdorff, then no strictly finer topology on X is compact and no strictly coarser topology on X is Hausdorff.*

Model solution. (a) We show that f is a closed map. Let $C \subset X$ be closed. Since X is compact, C is compact. Then $f(C)$, the continuous image of a compact set, is compact in Y. Since Y is Hausdorff, the compact subset $f(C)$ is closed in Y. Hence f is a closed map. Since f is a bijection, closedness of f is exactly continuity of f^{-1}: for C closed in X, $(f^{-1})^{-1}(C) = f(C)$ is closed in Y. Therefore f^{-1} is continuous and f is a homeomorphism.

(b) The coordinate functions cos and sin are continuous, so f is continuous. It is bijective: each point of S^1 is $(\cos t, \sin t)$ for a unique $t \in [0, 2\pi)$. However, f^{-1} is not continuous at $(1, 0)$. Indeed, for $t_n = 2\pi - 1/n$, the points $f(t_n) = (\cos t_n, \sin t_n)$ converge to $(1, 0)$ in S^1, but $f^{-1}(f(t_n)) = t_n$ does not converge to $f^{-1}((1, 0)) = 0$ in $[0, 2\pi)$. (These are metric spaces, so continuity can be tested with sequences.) Hence f is not a homeomorphism. Alternatively: S^1 is compact while $[0, 2\pi)$ is not, so no homeomorphism between them can exist.

Additional acceptable proof for Question 5(b): connectedness.

For $0 < \varepsilon < \pi$, the arc $A = \{(\cos t, \sin t) : -\varepsilon \le t \le \varepsilon\} \subseteq S^1$ is connected, whereas $f^{-1}(A) = [0, \varepsilon] \cup [2\pi - \varepsilon, 2\pi)$ is disconnected. Therefore f^{-1} cannot be continuous.

Alternative proof via failure to be an open map (rigor required).

Take $U = [0, \pi/2)$, open in $[0, 2\pi)$, and let $p = (1, 0) \in f(U)$. For any neighborhood N of p in S^1, choose an open $V \subseteq \mathbb{R}^2$ with $p \in V$ and $V \cap S^1 \subseteq N$. Then $B_{\mathbb{R}^2}(p, r) \subseteq V$ for some $r > 0$. For sufficiently small $\theta > 0$, $q = (\cos\theta, -\sin\theta)$ belongs to $N \cap S^1$ and to the fourth-quadrant arc, but $q \notin f(U)$. Hence no neighborhood of p in S^1 is contained in $f(U)$: p is not an interior point of $f(U)$, so f is not open.

Grading instruction — deduct 2 points.

If a non-open-map argument does not rigorously establish this neighborhood/subspace-topology claim, deduct 2 points uniformly in part (b). Do not apply this deduction to a complete, valid proof using another method.

(c) Let $X = \{0, 1\}$ with the discrete topology and $Y = \{0, 1\}$ with the trivial topology $\{\emptyset, Y\}$. Then X is compact (it is finite), Y is not Hausdorff (the two points have no disjoint neighborhoods), and the identity map $f\colon X \to Y$ is a continuous bijection: every function into a trivial space is continuous. But f^{-1} is not continuous, since $\{0\}$ is open in X while $(f^{-1})^{-1}(\{0\}) = \{0\}$ is not open in Y. Hence f is not a homeomorphism. (Any correct example, e.g., one with a cofinite target, is acceptable.)

(d) The identity map $\iota\colon (X, \mathscr{T}_2) \to (X, \mathscr{T}_1)$ is a bijection, and it is continuous because $\mathscr{T}_1 \subset \mathscr{T}_2$: the preimage of a $\mathscr{T}_1$-open set is $\mathscr{T}_2$-open. Its domain is compact and its codomain is Hausdorff, so by part (a) it is a homeomorphism. Then ι^{-1} is continuous, which says $\mathscr{T}_2 \subset \mathscr{T}_1$; hence $\mathscr{T}_1 = \mathscr{T}_2$.

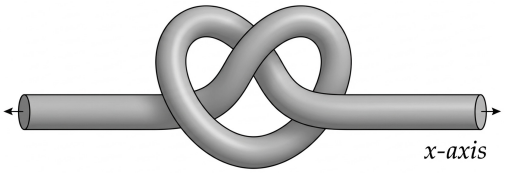
For the conclusion, suppose $(X, \mathscr{T})$ is compact Hausdorff. If $\mathscr{T}' \supsetneq \mathscr{T}$ were compact, applying the previous argument to $\mathscr{T} \subset \mathscr{T}'$ would give $\mathscr{T}' = \mathscr{T}$, a contradiction. If $\mathscr{T}' \subsetneq \mathscr{T}$ were Hausdorff, applying it to $\mathscr{T}' \subset \mathscr{T}$ would likewise give $\mathscr{T}' = \mathscr{T}$.

Suggested rubric. (a) 5 points: 1 for closed subsets of compact being compact, 1 for continuous images of compact being compact, 1 for compact subsets of Hausdorff being closed, 1 for assembling the closed-map argument, 1 for deducing continuity of f^{-1} from bijectivity and closedness. (b) 5 points: 1 for continuity, 1 for bijectivity, 3 for a rigorous proof that f^{-1} fails to be continuous (sequence argument or the compactness obstruction). (c) 5 points: 3 for a correct example, 2 for verifying the inverse is not continuous. (d) 10 points: 5 for applying part (a) to the identity map, 2 for concluding $\mathscr{T}_1 = \mathscr{T}_2$, 3 for both halves of the final conclusion.

Accept / watch for. In (a), all three compactness facts are needed; skipping "compact subsets of Hausdorff spaces are closed" is the most common gap. In (b), it is not enough to exhibit the oscillation or say the endpoints do not match; the discontinuity of f^{-1} at $(1, 0)$ must be argued (or compactness must be invoked). In (d), the identity map is continuous in only one direction; applying (a) to the wrong direction earns no credit.

Question 6 (10 points)

Problem. *A* long knot *is a subspace of* $\mathbb{E}^3$ *that is homeomorphic to* $\mathbb{E}^1$ *and agrees with the* x*-axis outside some ball. Let* T *be the solid tube of radius* 1 *around a long knot, as pictured below. You may assume that* T *is closed in* $\mathbb{E}^3$ *and that there is a homeomorphism* $h\colon T \to D^2 \times \mathbb{E}^1$*, where* $D^2 = \{(x, y) \in \mathbb{E}^2 \mid x^2 + y^2 \leqslant 1\}$.



A "thick" long knot T*: the two ends of the tube continue indefinitely.*

1. *(5 points) Show that the solid cylinder* $D^2 \times \mathbb{E}^1$ *is a retract of* $\mathbb{E}^3$*: construct a continuous* $\rho\colon \mathbb{E}^3 \to D^2 \times \mathbb{E}^1$ *with* $\rho(p) = p$ *for all* $p \in D^2 \times \mathbb{E}^1$.
2. *(5 points) Prove that there is a continuous* $r\colon \mathbb{E}^3 \to T$ *with* $r(t) = t$ *for all* $t \in T$.

Model solution. (a) Define $\rho\colon \mathbb{E}^3 \to D^2 \times \mathbb{E}^1$ by $\rho(x, y, z) = (\sigma(x, y), z)$, where $\sigma(x, y) = (x, y)$ if $x^2 + y^2 \leqslant 1$ and $\sigma(x, y) = (x, y)/\sqrt{x^2 + y^2}$ if $x^2 + y^2 \geqslant 1$. On the two closed sets $\{x^2 + y^2 \leqslant 1\} \times \mathbb{E}^1$ and $\{x^2 + y^2 \geqslant 1\} \times \mathbb{E}^1$, the map ρ is continuous, and the two formulas agree on their intersection $S^1 \times \mathbb{E}^1$. By the pasting lemma ρ is continuous, and by construction $\rho|_{D^2 \times \mathbb{E}^1} = \mathrm{id}$.

(b) Write $h = (h_1, h_2, h_3)\colon T \to D^2 \times \mathbb{E}^1 \subset \mathbb{E}^3$ with coordinate functions $h_i\colon T \to \mathbb{E}^1$. Since $\mathbb{E}^3$ is a metric space, it is T_4, and T is closed in $\mathbb{E}^3$; hence each h_i extends to a continuous $H_i\colon \mathbb{E}^3 \to \mathbb{E}^1$. Then $H = (H_1, H_2, H_3)\colon \mathbb{E}^3 \to \mathbb{E}^3$ is continuous and $H|_T = h$. With ρ from part (a), the composition $\rho \circ H\colon \mathbb{E}^3 \to D^2 \times \mathbb{E}^1$ is a continuous extension of h. Finally set

$$r = h^{-1} \circ \rho \circ H\colon \mathbb{E}^3 \to T.$$

This is continuous, and for $t \in T$ we have $r(t) = h^{-1}(\rho(H(t))) = h^{-1}(\rho(h(t))) = h^{-1}(h(t)) = t$, since $h(t) \in D^2 \times \mathbb{E}^1$. Hence r is a retraction of $\mathbb{E}^3$ onto T.

Suggested rubric. (a) 5 points: 2 for the radial formula, 2 for the pasting argument, 1 for checking $\rho|_{D^2 \times \mathbb{E}^1} = \mathrm{id}$. (b) 5 points: 1 for passing to coordinate functions, 1 for noting $\mathbb{E}^3$ is T_4 and T is closed, 1 for extending each h_i, 1 for inserting ρ to land in $D^2 \times \mathbb{E}^1$, 1 for the final composition $h^{-1} \circ \rho \circ H$ and the retraction check.

Accept / watch for. In (b), the extension theorem produces a map into $\mathbb{E}^1$, not into $D^2 \times \mathbb{E}^1$; without composing with ρ, the extension H may take values outside the cylinder, and $h^{-1} \circ H$ would not be defined. The whole point is that the argument never uses how the tube is knotted — only that T is closed and homeomorphic to $D^2 \times \mathbb{E}^1$.