Homework Problem: Adic Topologies, Inverse Limits, and Compactness

Undergraduate point-set topology, with a view towards real applications in research

Prerequisites. Basic point-set topology, including product topologies, closed subsets, and Tychonoff's theorem. All rings below are commutative with identity.

Goal. *Define the I-adic topology, relate it to inverse limits of finite discrete spaces, and use this relationship to prove that several formal-power-series rings are compact and Hausdorff.*

Step 1: Define the I-adic topology

Let R be a ring and let $I \subseteq R$ be an ideal. Define powers of I by

$$I^0 = R, \qquad I^1 = I, \qquad I^{n+1} = I^n I.$$

For $x \in R$ and $n \geq 0$, consider the coset

$$x + I^n = \{x + r \mid r \in I^n\}.$$

The *I-adic topology* on R is the topology generated by the basis

$$\mathcal{B}_I = \{x + I^n \mid x \in R,\ n \geq 0\}.$$

Intuitively, two elements are close if their difference lies in a high power of I.

(a) Prove that $\mathcal{B}_I$ is indeed a basis for a topology.

(b) Prove that, for every $x \in R$, the translation map

$$t_x : R \longrightarrow R, \qquad t_x(y) = x + y$$

is a homeomorphism in the I-adic topology.

(c) Prove that

$$\overline{\{0\}} = \bigcap_{n \geq 0} I^n.$$

Conclude that R is Hausdorff in the I-adic topology if and only if

$$\bigcap_{n \geq 0} I^n = 0.$$

Remark. *When $\bigcap_n I^n = 0$, define*

$$v_I(x) = \sup\{n \geq 0 \mid x \in I^n\} \qquad \textit{and} \qquad d_I(x, y) = 2^{-v_I(x-y)}.$$

Show that d_I is a metric inducing the I-adic topology.

Step 2: A compactness criterion using inverse limits

Let R be a ring and I an ideal. Suppose that

$$R/I^n$$

is finite for every $n \geq 1$, and give each quotient the discrete topology.

Define

$$\widehat{R}_I = \varprojlim_n R/I^n = \left\{ (r_n)_{n\geq 1} \in \prod_{n\geq 1} R/I^n \,\middle|\, r_{n+1} \equiv r_n \pmod{I^n} \text{ for every } n \right\}.$$

Give $\widehat{R}_I$ the subspace topology from the product topology.

(a) Show that $\widehat{R}_I$ is a closed subset of

$$\prod_{n\geq 1} R/I^n.$$

(b) Using Tychonoff's theorem, prove that $\widehat{R}_I$ is compact.

(c) Prove that $\widehat{R}_I$ is Hausdorff.

(d) Explain why the natural map

$$R \longrightarrow \widehat{R}_I$$

is continuous. What additional conditions ensure that it is injective or bijective?

The space $\widehat{R}_I$ is called the *I-adic completion* of R.

Step 3: Apply the criterion to $\mathbb{F}_p[\![a]\!]$

Fix a prime p, and let

$$\mathbb{F}_p[\![a]\!] = \left\{ \sum_{j=0}^{\infty} c_j a^j \,\middle|\, c_j \in \mathbb{F}_p \right\}$$

be the ring of formal power series over $\mathbb{F}_p$. Addition and multiplication are defined coefficientwise and by the usual Cauchy product formula.

Let

$$I = (a) = a\mathbb{F}_p[\![a]\!].$$

(a) Describe I^n explicitly.

(b) Prove that

$$\mathbb{F}_p[\![a]\!]/I^n \cong \mathbb{F}_p[a]/(a^n).$$

Conclude that these quotients are finite.

(c) Given a compatible sequence

$$(r_n) \in \varprojlim_n \mathbb{F}_p[\![a]\!]/I^n,$$

show that there is a unique formal power series

$$f = \sum_{j=0}^{\infty} c_j a^j$$

whose truncation modulo a^n equals r_n for every n.

(d) Deduce that

$$\mathbb{F}_p[\![a]\!] \cong \varprojlim_n \mathbb{F}_p[\![a]\!]/I^n.$$

(e) Conclude that $\mathbb{F}_p[\![a]\!]$, with its a-adic topology, is compact and Hausdorff.

Step 4: Extend the argument to $\mathbb{Z}_p[\![a]\!]$

Recall the description of the p-adic integers as an inverse limit:

$$\mathbb{Z}_p \cong \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}.$$

Thus an element of $\mathbb{Z}_p$ is a compatible sequence of residue classes modulo increasing powers of p. Give $\mathbb{Z}_p$ the corresponding inverse-limit topology.

Now let

$$R = \mathbb{Z}_p[\![a]\!]$$

and let

$$\mathfrak{m} = (p, a)$$

be the ideal generated by p and a.

(a) Show that

$$\mathfrak{m}^n = \left(p^n, p^{n-1}a, \ldots, pa^{n-1}, a^n\right).$$

(b) Explain why, modulo $\mathfrak{m}^n$, the coefficient of a^k is determined modulo

$$p^{n-k}\mathbb{Z}_p$$

for $0 \le k < n$.

(c) Conclude that

$$R/\mathfrak{m}^n$$

is finite for every n.

(d) **(Optional.)** Prove that

$$|R/\mathfrak{m}^n| = p^{1+2+\cdots+n} = p^{n(n+1)/2}.$$

(e) Show that a compatible sequence in

$$\varprojlim_n R/\mathfrak{m}^n$$

determines each coefficient of a as a compatible sequence modulo powers of p, and therefore determines an element of $\mathbb{Z}_p[\![a]\!]$.

(f) Conclude that

$$\mathbb{Z}_p[\![a]\!] \cong \varprojlim_n \mathbb{Z}_p[\![a]\!]/\mathfrak{m}^n.$$

(g) Prove that $\mathbb{Z}_p[\![a]\!]$, equipped with its $\mathfrak{m} = (p, a)$-adic topology, is compact and Hausdorff.

Discussion questions

1. Where is the hypothesis that each quotient R/I^n is finite used?
2. Why is the inverse limit a closed subset of the product?
3. What happens if $\bigcap_n I^n \neq 0$?
4. Why does completeness alone not imply compactness? What additional property of the finite quotients is essential here?