MA323 Topology Midterm Exam

2:00–3:50 pm

April 18, 2025

Your name: __________________ SID: __________________

1. (15 points)

   (a) (5 points) State the definition of a T_3 space.

   (b) (10 points) Show that a compact Hausdorff space is T_3.

2. (20 points)

   (a) (5 points) State the definition of a C_2 space.

   (b) (5 points) Show that a C_2 space is separable.

   (c) (10 points) Give an example of a separable space that is not C_2, with justification.

3. (15 points) Give examples with justification.

   (a) (5 points) A closed and bounded metric space that is not compact.

   (b) (5 points) A pair of homeomorphic metric spaces, one complete, the other not (recall that by definition a metric space is *complete* if every Cauchy sequence in it converges).

   (c) (5 points) A continuous bijection whose inverse is not continuous.

4. (20 points) Let $f\colon X \to Y$.

(a) (5 points) Show that if f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$.

(b) (5 points) State the definition of a C_1 space.

(c) (10 points) Show that the converse of (a) holds if X is C_1.

5. (20 points) Recall that a topology $\mathscr{T}$ can be specified by a *subbasis* $\mathscr{S}$, so that its basis $\mathscr{B}$ consists of all finite intersections of elements of $\mathscr{S}$, and $\mathscr{T}$ consists of all arbitrary unions of elements of $\mathscr{B}$.

   Let X and Y be topological spaces. Denote by $\mathscr{C}(X,Y)$ the set of continuous maps from X to Y. Given any $C \subset X$ compact and $U \subset Y$ open, write

$$S(C,U) = \{f \in \mathscr{C}(X,Y) \mid f(C) \subset U\}$$

   Define the *compact-open topology* on $\mathscr{C}(X,Y)$ to be generated by the subbasis $\mathscr{S}$ consisting of sets of the form $S(C,U)$.

   (a) (5 points) Check that $\mathscr{S}$ is indeed a subbasis.

   (b) (15 points) Given a third space Z, define a map $\phi\colon \mathscr{C}(X \times Y, Z) \to \mathscr{C}\big(X, \mathscr{C}(Y,Z)\big)$ by sending $f\colon X \times Y \to Z$ to $F\colon X \to \mathscr{C}(Y,Z)$ such that each $x \in X$ maps to

$$\begin{aligned} F(x)\colon Y &\to Z \\ y &\mapsto f(x,y) \end{aligned}$$

   Check that ϕ is well-defined, i.e., each $F(x)$ is continuous and F is also continuous. (Hint: write $F(x)$ as a composite of continuous maps. To show the continuity of F, for each fixed $x_0 \in X$, you need only consider neighborhoods of $F(x_0)$ that are in the subbasis $\mathscr{S}$. Use compactness in the definition of $\mathscr{S}$.)

6. (10 points) If X and Y are topological spaces, a map $f: X \to Y$ (continuous or not) is said to be *proper* if the preimage of each compact subset of Y is compact. Show that if f is an embedding with closed image, then f is proper.