Averages:

#1	#2	#3	#4	#5	#6	#7	Total
7.3/10	8.6/15	7.8/10	6.0/20	11.5/15	8.7/15	9.7/15	59.6/100

Distribution: median = 61

90–100: x

80–89:

70–79: x x x x x

60–69: x||x x x x x

50–59: x x x x x

40–49: x x x

<40: x x

MA323 Topology Midterm Exam

2:00–3:50 pm

April 19, 2024

Your name: ________________ SID: ________________

1. (10 points) Answer the following questions. No justification is needed.

   (a) (5 points) True or false? Any compact subspace of a Hausdorff space is closed.

   True.

   (b) (5 points) Recall that a metric space is said to be *complete* if any Cauchy sequence in it converges. Give an example of two homeomorphic metric spaces such that one of them is complete whereas the other is not.

   The real line with the Euclidean topology, and a finite open interval with the subspace topology.

2. (15 points) True or false? If true, give a proof. If false, give a counterexample.

   (a) (5 points) Any locally path-connected space is path connected.

   False. The 1-dimensional Euclidean space with a point removed is a locally path-connected space that is not path connected.

   (b) (5 points) Any path-connected space is locally path connected.

   False. Consider the subspace

   $$(\mathbb{Q} \times \mathbb{R}) \cup (\mathbb{R} \times \{0\})$$

   of the Euclidean plane. It is path connected (the horizontal real line serves as a "tunnel") but not locally path connected (e.g., consider neighborhoods of the point $(0,1)$).

   (c) (5 points) Let X and Y be topological spaces. If X is compact, the projection map $p\colon X \times Y \to Y$ is closed.

   True. Given any closed subspace F of $X \times Y$, to show $p(F)$ is closed in Y, we may assume that $p(F)$ is not the entire Y, so that it suffices to prove that any point of Y in the complement is an interior point. Let y be such a point. Then $X \times \{y\}$ does not intersect F. Since F is closed, for each x in X, there are an open neighborhood U_x of x and an open neighborhood V_x of y such that $U_x \times V_x$ does not intersect F. By compactness of X, the open cover $\{U_x\}$ has a finite subcover $\{U_{x_i}\}$. Taking the intersection of the corresponding open sets V_{x_i}, we then obtain the desired open neighborhood of y away from $p(F)$.

3. (10 points) Let $f: X \to Y$ be a surjective continuous map of topological spaces.

(a) (5 points) State the definition of f being a quotient map.

The map f is a quotient map if any subset V of Y with an open preimage in X is open in Y.

(b) (5 points) Suppose that X is compact and Y is Hausdorff. Show that f is a quotient map.

Let V be a subset of Y as above. We need to show that it is open. Consider instead its complement $F = Y \setminus V$. The preimage of F in X is then closed and hence compact, since X is compact. Because f is a continuous surjection, F is compact and hence closed, since Y is Hausdorff. Therefore V is open.

4. (20 points) Recall that a *topological group* is a group G endowed with a topology such that the group multiplication and taking inverse are continuous operations, i.e., the maps $G \times G \to G$, $(g_1, g_2) \mapsto g_1 g_2$ and $G \to G$, $g \mapsto g^{-1}$ are continuous.

   Consider the group $\mathrm{SL}_2(\mathbb{R})$ of all 2×2 real matrices with determinant one, with the topology induced from the coordinate embedding into $\mathbb{E}^4$.

   (a) (5 points) Prove that it is a topological group.

   Since matrix multiplication and inversion both involve only algebraic functions of the four entries which are continuous, it is not hard to check that the group multiplication and taking inverse here are continuous operations with respect to the topology on $SL_2(\mathbb{R})$ inherited from the 4-dimensional Euclidean space.

   (b) (5 points) Show that its subgroup $\mathrm{SL}_2(\mathbb{Z})$ of matrices with integral entries is locally compact.

   We observe that with each entry an integer instead of a real number, the subgroup $SL_2(\mathbb{Z})$ consisting of such matrices is discrete, i.e., any point has a neighborhood (in the subspace topology) consisting of solely that point, which is clearly compact.

(c) (10 points) Show that the torus $T = S^1 \times S^1$ is also a topological group and that $\mathrm{SL}_2(\mathbb{Z})$ acts continuously on it.

Observe that $\mathcal{T}$ is homeomorphic to the quotient space of the Euclidean plane modulo the integral lattice $\mathcal{Z} \times \mathcal{Z}$. As such, it inherits continuous group operations and an $\mathcal{SL}_2(\mathcal{Z})$-action from the Euclidean plane, since they preserve the lattice.

5. (15 points) A map $f : X \to Y$ is *locally constant* if for each $x \in X$ there is an open set U with $x \in U$ and $f|_U$ constant. Prove or disprove: if X is connected and Y is any space, then every locally constant map is constant.

True. If not, f(x) differs from f(x') for some x and x' in X. Then there exist open subsets U and U' containing x and x' respectively, over each of which f restricts to be constant. Clearly, U and U' must be disjoint, which contradicts the connectedness of X.

6. (15 points) Suppose that (X, d) is a compact metric space and $f\colon X \to X$ is an isometry, i.e., $d\big(f(x), f(y)\big) = d(x, y)$ for all $x, y \in X$. Show that f is a homeomorphism.

Since X is both compact and Hausdorff, we need only show that f is a continuous bijection. The continuity follows by definition since f is an isometry. So does injectivity. To show surjectivity, suppose x is not in the image. Since X is compact and f is continuous, $f(X)$ is compact. Since X is Hausdorff, $f(X)$ is closed. Thus there exists an open neighborhood $\mathcal{U}$ of x contained in $X \setminus f(X)$. Since $\mathcal{U}$ and $f(\mathcal{U})$ do not intersect, neither do $f(\mathcal{U})$ and $f(f(\mathcal{U}))$. Iterating this process, we obtain infinitely many disjoint subsets in $f(X)$, all isometric to $\mathcal{U}$. This leads to a contradiction as $f(X)$ is compact.

7. (15 points) Is there a continuous injective function $f : S^1 \to \mathbb{E}^1$? If yes, give an example. If no, give a proof.

No. If yes, since f has a compact source and Hausdorff target, its image would be homeomorphic to the circle. If we remove a point from this subspace, the remaining points still belong to a single path component. However, this is not the case when this point sits on the real line, a contradiction.