

## Assignment 9

1. [Y] Sec. 3.3 #2.
2. [Y] Sec. 3.4 #2.
3. Show that the map  $\mathbb{E}^3 \rightarrow \mathbb{E}^4$  given by  $(x, y, z) \mapsto (xy, xz, y^2z^2, 2yz)$  induces an embedding of the projective plane  $\mathbb{R}\mathbb{P}^2$  into  $\mathbb{E}^4$ .

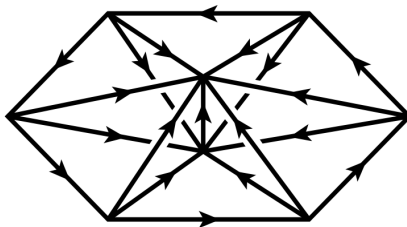
Analogous to the projective plane, the 3-dimensional real projective space  $\mathbb{R}\mathbb{P}^3$  can be obtained by identifying antipodal points of the 3-sphere  $S^3$ , or by identifying antipodal points on the boundary  $S^2$  of the solid 3-ball  $D^3$ , or by equipping the set of 1-dimensional subspaces of  $\mathbb{R}^4$  with a suitable topology.

4. This last description gives  $\mathbb{R}\mathbb{P}^3$  the structure of a 3-manifold as follows. Each point of  $\mathbb{R}\mathbb{P}^3$  has a *homogeneous coordinate*  $(x_0 : x_1 : x_2 : x_3)$ , i.e., the points  $(x_0, x_1, x_2, x_3)$  and  $(\lambda x_0, \lambda x_1, \lambda x_2, \lambda x_3)$  of  $\mathbb{R}^4 - \{(0, 0, 0, 0)\}$  are identified as the same point in  $\mathbb{R}\mathbb{P}^3$  for any nonzero real number  $\lambda$ .

Based on this, specify the local charts that cover  $\mathbb{R}\mathbb{P}^3$ , each of which is homeomorphic to  $\mathbb{E}^3$ , and write down the transition functions on their pairwise overlaps.

In the following, let us give two more descriptions for  $\mathbb{R}\mathbb{P}^3$ , first as a *lens space* (introduced by Tietze) with the structure of a simplicial complex (see, e.g., Section 3.2 of the reference [B]), second as a quotient space obtained by a *Dehn surgery*.<sup>1</sup>

5. Construct a 3-dimensional simplicial complex from  $n$  tetrahedra (i.e., 3-simplices)  $T_1, \dots, T_n$  by the following two steps. First arrange the tetrahedra in a cyclic pattern as in the figure, so that each  $T_i$  shares a common vertical face with its two neighbors  $T_{i-1}$  and  $T_{i+1}$ , subscripts being taken mod  $n$ . Then identify the bottom face of  $T_i$  with the top face of  $T_{i+1}$  for each  $i$ .



This simplicial complex, or its polytope (geometric realization), is an example of a *lens space*, denoted by  $L(n, 1)$ .

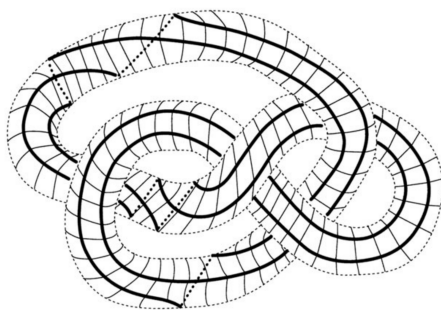
- (a) Show that  $L(2, 1)$  is homeomorphic to  $\mathbb{R}\mathbb{P}^3$ .
- (b) Calculate the Euler characteristic of  $\mathbb{R}\mathbb{P}^3$  by carefully enumerating the simplices of  $L(2, 1)$ .

<sup>1</sup>The figures are copied from Allen Hatcher's *Algebraic topology* and John Luecke's *Dehn surgery on knots in the 3-sphere*. The descriptions below are adapted in addition from Joshua Evan Greene's *Heegaard Floer homology*.

6. More generally, viewing  $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$  and given positive integers  $p, q$  with  $(p, q) = 1$ , we can construct the lens space  $L(p, q)$  from the periodic homeomorphism  $f: S^3 \rightarrow S^3, (z_1, z_2) \mapsto (e^{2\pi i/p} z_1, e^{2\pi i q/p} z_2)$  as the quotient space  $S^3 / \sim_f$ , where  $x \sim_f x'$  if and only if the  $k$ -fold composite  $f^k(x) = x'$  for some  $k$ .

Show that this construction of  $L(p, 1)$  gives the same space as Question 5.<sup>2</sup>

In 1910, Dehn devised a general method called *surgery* for constructing 3-manifolds, which can also be carried out in two steps as follows. Dehn's construction begins with a knot  $K \subset S^3$ , i.e., an embedded  $S^1 \hookrightarrow S^3$ . A closed tubular neighborhood of  $K$  is homeomorphic to a solid torus  $S^1 \times D^2$ . First we excise its interior from  $S^3$  to produce the knot exterior  $X_K$ , a compact manifold with torus boundary. We then obtain a 3-manifold by regluing a solid torus  $S^1 \times D^2$  to  $X_K$  along their boundaries, in such a way that a curve  $\{\theta\} \times \partial D^2$  (i.e., a line of longitude) glues to a curve that wraps  $p$  times longitudinally and  $q$  times meridionally around  $K$ .



The homeomorphism type of the result depends only on  $K$  and the slope  $p/q$ , and we denote it  $K(p/q)$ .

7. Let  $K = \bigcirc$  be the unknot. Show that  $\bigcirc(p/1) \cong L(p, 1)$  and so Dehn's surgery gives yet another way of constructing  $\mathbb{RP}^3$ , when  $p = 2$ .

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<sup>2</sup>To visualize  $S^3$  from  $D^3$ , consider the analogue of  $S^2$  as obtained from  $D^2$  by folding a dumpling, i.e., by identifying pairs of points on the boundary  $\partial D^2 = S^1$  that are symmetric along a diameter  $D^1$ . It is also helpful to think of  $S^3$  as the union of a pair of linked solid tori, by drilling off a solid cylinder through the north and south poles of  $D^3$ .