Supplementary reading III:

Function space, the compact-open topology, and homotopy of maps

Let X be a set and (Y,d) be a metric space.

Write $Y^X :=$ the set of functions from X to Y

(cf notation 2^X of the power set and X^I of the product of copies of X indexed by I)

Define the uniform metric on Y^X corresponding to d by

$$\bar\rho(f,g) := \sup\{\bar d(f(x),g(x)) \mid x\in X\}$$

for any $f, g : X \longrightarrow Y$, where $\bar d(a,b) = \min\{d(a,b), 1\}$ is the standard bounded metric associated to d.

Then
$$\Big(\underset{\text{continuous}}{\mathcal{C}}(X,Y),\bar\rho\Big) \underset{X\text{ compact}}{\overset{\text{closed}}{\subset}} \Big(\underset{\text{bounded}}{\mathcal{B}}(X,Y),\bar\rho\Big) \overset{\text{closed}}{\subset} \big(Y^X,\bar\rho\big)$$

Recall that a topology $\mathcal{T}$ can be specified by a subbasis $\mathcal{S}$ so that it basis $\mathcal{B}$ consists of all finite intersections of elements of $\mathcal{S}$, and $\mathcal{T}$ consists of all arbitrary unions of elements of $\mathcal{B}$.

Def (compact-open topology) Let X and Y be topological spaces (Y not necessarily metrizable). Given $C \underset{\text{compact}}{\subset} X$ and $U \underset{\text{open}}{\subset} Y$, write

$$\underset{\text{subbasis}}{S(C,U)} = \{f\in\mathcal{C}(X,Y) \mid f(C)\subset U\}$$

Define the compact-open topology on $\mathcal{C}(X,Y)$ to be by the subbasis (check!) consisting of sets of the form $S(C,U)$.

Note This topology is useful and important (later).

Why compact in domain and open in range?

compact: a finite condition

open: measure closeness (ε-balls)
↑
no-ed

Compare: ① point-open topology generated by

$$\mathcal{S} = \{S(x,U) \mid \underset{\text{just a set}}{x\in X},\ \underset{\text{open}}{U\subset Y}\}$$

= topology of pointwise convergence — not topological (related to "completeness" compactness (equicontinuous))

= the product topology on Y^X

Problem $\mathcal{C}(X,Y)\subset Y^X$ is not closed under this topology (pointwise convergence $\not\Rightarrow$ uniform convergence).

② topology of compact convergence

for $(X,\mathcal{T})$, (Y,d) generated by

$$\mathcal{B} = \{B_C(f,\varepsilon) \mid \underset{\text{compact}}{C\subset X},\ f\in Y^X,\ \varepsilon>0\}$$

$f_n \to f \iff f_n|_C \to f|_C$ in the uniform topology for each compact C

$\mathcal{C}(X,Y)\subset Y^X$ is closed under this topology if X is compactly generated, i.e., A is open/closed in X if $A\cap C$ is open/closed in C for each compact $C\subset X$. Ex: locally compact spaces, C_1 spaces

On Y^X, Y metric

$$(\text{uniform}) \underset{X \text{ compact}}{\supseteq} (\text{compact convergence}) \underset{X \text{ discrete}}{\supseteq} (\text{pointwise convergence})$$

$\|$ On $\mathcal{C}(X,Y)$

(compact-open)

Summary

X compactly generated (e.g. locally compact spaces, C_1 spaces)

- Topologies for function spaces $\mathcal{C}(X,Y) \underset{\text{closed}}{\subset} \mathcal{B}(X,Y) \subset Y^X$

✓ compact-open topology $\mathcal{S} = \{ S(C,U) \mid C \text{ compact in } X, U \text{ open in } Y \}$

↑ subbasis (check!)

× point-open topology / topology of pointwise convergence $\mathcal{S} = \{ S(x,U) \}$

pointwise convergence $\not\Rightarrow$ uniform convergence

✓ topology of compact convergence $\mathcal{B} = \{ B_C(f,\varepsilon) \mid C \text{ compact in } X, f \in Y^X, \varepsilon > 0 \}$

✓ uniform topology $\mathcal{B} = \{ B_{\bar{\rho}}(f,\varepsilon) \}$

Observe Given spaces X, Y, Z, any continuous function $f : X \times Y \to Z$ induces a continuous function $F : X \to \mathcal{C}(Y,Z)$ → equipped with compact-open topology

$$x \longmapsto F(x) : Y \to Z$$
$$y \mapsto f(x,y)$$

and conversely any continuous function $G: X \to \mathcal{C}(Y, Z)$ induces a continuous function $g: X \times Y \to Z$

$$(x, y) \mapsto (G(x))(y)$$

Thm Suppose X, Y, Z are compactly generated and weakly Hausdorff (any continuous function into X sends a compact Hausdorff space onto a closed subspace of X). Then there is a homeomorphism

$$\mathcal{C}(X \times Y, Z) \cong \mathcal{C}(X, \mathcal{C}(Y, Z)).$$

(The subcategory of CGWH spaces is Cartesian closed.

Compare $\mathrm{Hom}_R(M \otimes_R N, P) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))$.

adjunction between the functors

$$\underset{\text{Left adjoint}}{- \otimes_R N} \quad \text{and} \quad \underset{\text{right adjoint}}{\mathrm{Hom}_R(N, -)}$$)

Ex / Def homotopy between continuous maps as a path in the space of continuous maps.

Given $p: I \longrightarrow \mathcal{C}(X, Y)$ with $p(0) = f$ and $p(1) = g$,

$$p \in \mathcal{C}(I, \mathcal{C}(X \times Y)) \cong \mathcal{C}(I \times X, Y) \cong \mathcal{C}(X \times I, Y)$$

$$p \longmapsto H$$

Given continuous maps $f, g: X \to Y$, if there exists a continuous map $H : X \times I \longrightarrow Y$ such that $H|_{X \times \{0\}} = f$

$H|_{X \times \{1\}} = g$, we then say that

$H(x, 0) = f(x), \forall x \in X$

$H|_{X \times \{t\}}$ a "slice"

f and g are homotopic, denoted $f \simeq g$, and call H a homotopy from f to g.

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