Supplementary reading II:

Simplicial complex, Euler characteristic, (orientation)

Simplicial complex (单纯复形)

Def Given points $A_0, \dots, A_n$ in E^N, if they satisfy

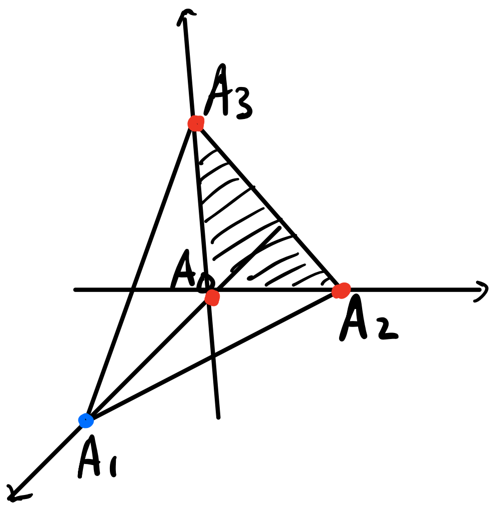
$$x_0 + \cdots + x_n = 0 \text{ and } x_0 A_0 + \cdots + x_n A_n = 0 \iff x_0 = \cdots = x_n = 0$$

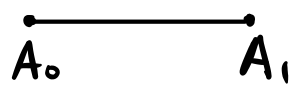
then we say that $A_0, \dots, A_n$ are in generic position.
处于一般位置

Note This is equivalent to the requirement that the n vectors $A_1 - A_0, \dots, A_n - A_0$ are linearly independent.

Def When $A_0, \dots, A_n$ are in generic position, the closed convex polytope $\{x_0 A_0 + \cdots + x_n A_n \mid x_0 + \cdots + x_n = 1,\ x_0, \dots, x_n \in [0,1]\}$ is called an n-simplex (n维单形).
The points $A_0, \dots, A_n$ are called the vertices (顶点) of this simplex.

If the vertices of a simplex t are all vertices of a simplex s, we call t a face (面) of s.





Note The condition that the vertices are in general position guarantees that each point in the simplex has a unique tuple of coordinates $(x_0, \dots, x_n)$.

Def A set K of simplices in E^N is called a simplicial complex (单纯复形) if

(1) $s \in K \Rightarrow$ each face of $s \in K$

(2) $s, t \in K$, $s \cap t \neq \phi \Rightarrow s \cap t \in K$

The highest dimension of a simplex in K is called the <u>dimension of K</u>.

$|K| := \text{"}\bigcup_{s \in K}\text{"} s$ is called the <u>polytope</u> of K.
多面体
/ geometric realization

(1895-1965, Hungarian)
<u>Radó's lemma</u> Each closed surface is homeomorphic to
1920s (not Radon-Nikodym)
the polytope of a finite 2-dimensional simplicial complex.

3-manifolds: Moise, Bing 1950s (Munkres, Smale, J.H.C. Whitehead admitting unique smooth structure)

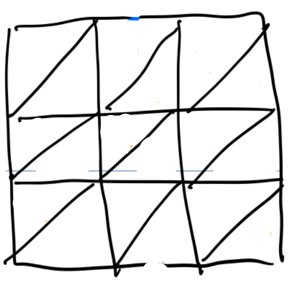
4-manifolds; E_8-manifold is not triangulable. Freedman 1982 (not admitting a smooth structure
some others admit infinitely many nonequivalent smooth structures)

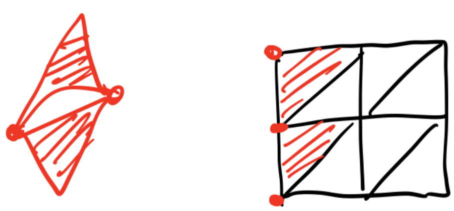
$\geqslant 5$-manifolds: existing non-triangulable manifolds in each dimension. Manolescu 2016

Wang, Xu 2017 S^{61} admits a unique smooth structure

TOP, DIFF, PL

<u>Ex</u> torus

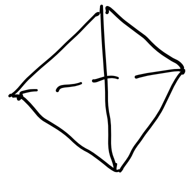
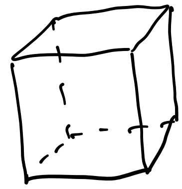




not satisfying (2) in definition of simplicial complexes

<u>Ex</u> projective plane?

Euler characteristic(given a convex polytope, # vertices − # edges + # faces = 2)

4−6+4 8−12+6

Define
$\chi(K) :=$ # even-dimensional simplices − # odd-dimensional simplices

characteristic

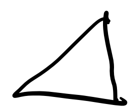
Prop When $|K| \cong gT^2$, $\chi(K) = 2-2g$.

When $|K| \cong kP^2$, $\chi(K) = 2-k$.

Note ① $\chi(K)$ is a topological invariant, called the Euler characteristic of $|K|$

歐拉(示性)數

② In fact, when $|K|$ is homeomorphic to a compact, connected surface with boundary, $\chi(K)$ is also determined by the homeomorphism class of $|K|$. For example, if $|K|$ is homeomorphic to a disc, $\chi(K) = 1$.



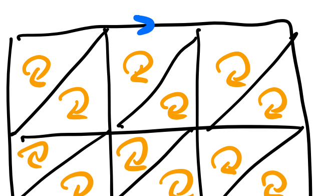
3−3+1

③ The proof uses homology. (# m-"holes" = the rank of $H_m(K)$ as a free ab group).

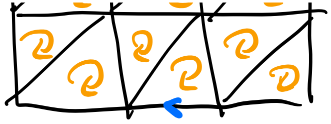
同調

Ex torus

orientable

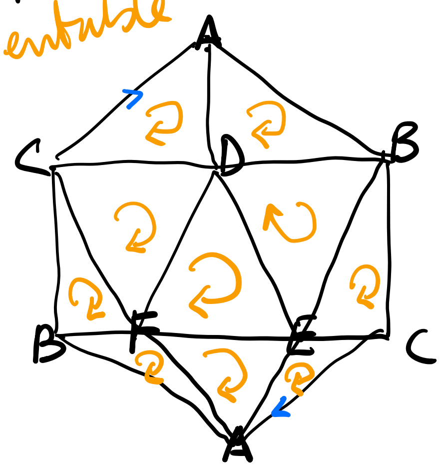


$9 - 27 + 18 = 0$



Ex projective plane

unorientable



"orientation"

Ex $\chi(M_1 \# M_2) = \chi(M_1) + \chi(M_2) - 2$

$-3 + 3 - 2$

Recognition of topological types of closed surfaces

Given 2-dimensional complexes K_1 and K_2, suppose their polytopes are closed surfaces. Then

$|K_1| \cong |K_2| \iff$ their Euler characteristics and orientability agree.

$$H_n(M) = \begin{cases} \mathbb{Z} & \text{orientable} \\ 0 & \text{nonorientable} \end{cases}$$

Note ① The Euler characteristics of odd-dimensional manifolds equal zero.

② 4-dimensional simply-connected closed manifolds

$M \cong N$

$\iff Q_M \cong Q_N$ intersection form $H^2(M;\mathbb{Z}) \times H^2(M;\mathbb{Z}) \to \mathbb{Z}$

$ks(M) = ks(N)$

Kirby-Siebenmann invariant

$ks(M) \in H^4(M;\mathbb{Z}/2) = \mathbb{Z}/2$

Q even: $\left(Q, \frac{\text{sign}(Q)}{8} \bmod 2\right)$ can be realized

divisible

not even: $(Q, \mathbb{Z}/2)$ can be realized

③ Computational topology, Edelsbrunner-Haren

e.g. orientability

Summary

$2-2g$ / $2-k$ Yes / No

· Euler characteristic and orientability as complete topological invariants for 2-manifolds.

(connected, closed)

Mentioned $(Q_M, ks(M))$ as those for 4-manifolds

(simply connected)

and realization problem: Q even, $\left(Q, \dfrac{\operatorname{sign}(Q)}{8} \bmod 2\right)$ divisible

Q not even, $(Q, 0 \text{ or } 1)$

are realizable by some 4-mfd.