Supplementary reading I:

Embeddings of manifolds (into Euclidean spaces)

Ex $S^2 \hookrightarrow \mathbb{E}^3$, $\mathbb{R}P^2 \hookrightarrow \mathbb{E}^4$

manifold structure / charts?
generalization

$\mathbb{R}P^2 \not\hookrightarrow \mathbb{E}^3$ (by Alexander duality, needs homology and cohomology)

Def Let $\{U_1, \ldots, U_k\}$ be a finite (can be more general, suffices here for applications) indexed open cover of the space X. An indexed family of continuous functions

$\varphi_i : X \longrightarrow [0,1]$ for $i = 1, \dots, k$

is said to be a partition of unity dominated by / subordinate to $\{U_i\}$ if

单位分解

- $\underline{\operatorname{supp}\phi_i} \subset U_i$ for each i

  $:= \overline{\{x \in X \mid \phi_i(x) \neq 0\}}$

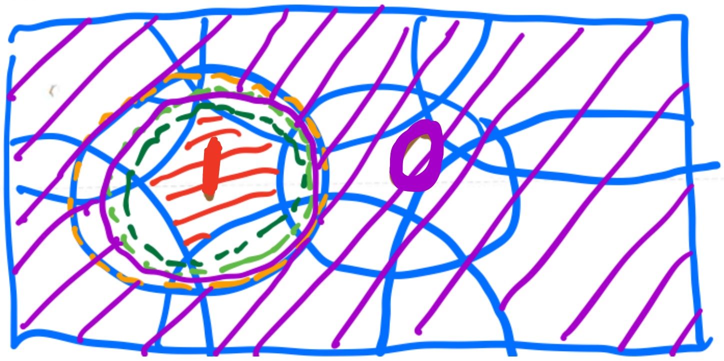
- $\sum_{i=1}^{k} \phi_i(x) = 1$ for each x

  $\uparrow$ unity

<u>Thm</u> If X is T_4, then a partition of unity exists for any finite open cover $\{U_1, \dots, U_k\}$ of X.

<u>Pf</u> A careful application of the Urysohn lemma.

(harder for general X and cover, paracompact)



 $\phi_1 > 0$

$$\tilde{\phi}_i = \frac{\phi_i}{\sum_{k=1}^{n} \phi_k} \qquad \square$$

$A_1 = X - U_2 - U_3 - \cdots - U_k$

$B_1 = X - U_1' \supset X - U_1$

$\Big\} \overset{\text{Urysohn}}{\rightsquigarrow} \phi_1 : X \longrightarrow [0,1] \rightsquigarrow U_1' = \phi_1^{-1}(0,1]$

$A_2 = X - U_1'' - U_3 - \cdots - U_k$

cpt + Hausdorff $\Rightarrow T_4$

Cor If X is [illegible] then X can be embedded

Cor If X is a cpt n-manifold, then X can [illegible] into $\mathbb{R}^N$ for some positive integer N.

Pf $X = \bigcup_{i=1}^{k} U_i$ with $f_i : U_i \xrightarrow{\cong} E^n$

Let $\{\phi_i\}_{i=1}^{k}$ be a partition of unity dominated by $\{U_i\}$. Write $A_i = \operatorname{supp} \phi_i$ and define

$$\tilde{f}_i(x) = \begin{cases} \phi_i(x) \cdot f_i(x) & x \in U_i \\ \vec{0} & x \notin X - A_i \end{cases}$$

(continuity is local on the source)

Define the embedding

$$f : X \longrightarrow \underbrace{E \times \cdots \times E}_{k \text{ times}} \times \underbrace{E^n \times \cdots \times E^n}_{k \text{ times}}$$

by

$$f(x) = (\underbrace{\phi_1(x), \ldots, \phi_k(x)}_{\text{for injectivity}}, \tilde{f}_1(x), \ldots, \tilde{f}_k(x)).$$

$\square$

Note paracompactness [M, Thm 41.7] See also [M, Thm 50.5] (sharper) and the Whitney embedding theorem for the smooth category.