#1 #2 #3 #4 #5 total

$14.3/20$ $17.6/20$ $11.6/20$ $14.9/20$ $11.8/20$ $70.3/100$

Averages

median = 74

90–100: × × × × × × × × ×

80–89: × × ×

70–79: × × × × × ×

74

60–69: × × × ×

50–59: × × × ×

<50: × × × × ×

MA323 Topology Midterm Exam

2:00–3:50 pm

November 17, 2023

Your name: ____________ SID: ____________

1. (20 points) For each of the following statements: if the statement is true, write "True." If the statement is not true, state a counterexample. No further justification is needed.

   (a) Let (X, d) be a metric space and $S \subset X$ be a finite set. Then $\mathring{S} = \emptyset$.

   Any set equipped with the discrete topology can be viewed as a metric space with metric d given by $d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$. Thus $\mathring{S} = S$.

   (b) Let (X, τ) be a topological space and $x \in X$. Then the set $\{x\} \subset X$ is closed.

   We need a space that is not T_1. Let $X = \{x, y\}$ with the trivial topology, i.e., the open sets are ϕ and X.

   (c) Suppose that (X, d) is a compact metric space. Then X is bounded.

   True.

   (d) Let (X, τ) be a compact topological space. Then every compact subset of X is closed.

   We need a space that is not T_2. Again, let $X = \{x, y\}$ with the trivial topology. It is compact because it is finite. The set $\{x\}$ is compact but not closed.

2. (20 points) Let X be a topological space and let $X \times X$ be the product space. The set

$$\Delta = \{(x, x) \mid x \in X\} \subset X \times X$$

is called the diagonal of $X \times X$. Prove that X is Hausdorff if and only if the diagonal Δ is a closed subset of $X \times X$.

"$\Rightarrow$": It suffices to show that $X \times X - \Delta$ is open. Let $(x, y) \in X \times X - \Delta$ so that $x \neq y$. Since X is Hausdorff, there exist open subsets U and V such that $U \ni x$, $V \ni y$, and $U \cap V = \phi$.

Thus $(x, y) \in U \times V \subset X \times X - \Delta$.

"$\Leftarrow$": Given $x, y \in X$ with $x \neq y$, since $X \times X - \Delta$ is open, there exists an open subset $W \subset X \times X$ such that $(x, y) \in W \subset X \times X - \Delta$. Since the product topology on $X \times X$ is generated by subsets of the form $U \times V$ where U and V are open subsets of X, there exist U_x and V_x open in X such that $(x, y) \in U_x \times V_x \subset W \subset X \times X - \Delta$. Thus $U_x \ni x$ and $V_x \ni y$ such that $U_x \cap V_x = \phi$.

3. (20 points) Let X and Y be topological spaces and $f : X \to Y$ be a function. Define the *graph* of f to be

$$\Gamma = \{(x, y) \in X \times Y \mid y = f(x)\}$$

Consider the following statement: if Γ is a closed subset of $X \times Y$, then f is continuous.

(a) If in addition Y is compact, prove the above statement.

Given any $x \in X$, let V be an open neighborhood of $f(x)$. Given any $y \neq f(x)$, since Γ is closed, there exist open $V_y \ni y$ and open $U_y \ni x$ such that $U_y \cap f^{-1}(V_y) = \phi$. Since Y is compact and $Y = \bigcup_{y \neq f(x)} V_y \cup V$, we have $Y = V \cup \bigcup_{i=1}^{n} V_{y_i}$ for some n. Let $U = \bigcap_{i=1}^{n} U_{y_i}$. Then $f(U) \subset V$.

(b) If in addition Y is Hausdorff, prove the converse of the above statement.

Let $(x, y) \in X \times Y - \Gamma$ so that $y \neq f(x)$. Since Y is Hausdorff, there exist open subsets U and V of Y such that $U \ni f(x)$, $V \ni y$, and $U \cap V = \phi$.

Since f is continuous, $f^{-1}(U)$ is open. Then $(x, y) \in f^{-1}(U) \times V \subset X \times Y - \Gamma$. Therefore $X \times Y - \Gamma$ is open and so Γ is closed.

4. (20 points) Give an example of a topological space X and a finite subset $A \subset X$ whose closure $\overline{A}$ is infinite. Is there an example if X is Hausdorff?

Let $X = \mathbb{R}$ equipped with the trivial topology and $A = \{0\}$. Then given any $x \in \mathbb{R}$, its only open neighborhood is $\mathbb{R}$, which intersects A. Thus $x \in \overline{A}$. Therefore $\overline{A} = \mathbb{R}$.

No. If X is Hausdorff, any singleton is closed, and hence so is any finite subset.

5. (20 points) Let $\{0,1\}$ denote the 2-element set with the discrete topology, and let $C = \{0,1\}^{\mathbb{N}}$ be the product of countably infinitely many copies of the 2-element set, with the product topology.

(a) Prove that C is sequentially compact.

Let $\{\vec{v}_n\}_{n=1}^{\infty} \subset C$ be a sequence.
The first components $v_{n,1}$ of $\vec{v}_n$ must have infinitely many 0 or 1, say 0.
Among these $\vec{v}_{n_k}$, their second components $v_{n_k,2}$ must have infinitely many 0 or 1, say 0. Inductively, we obtain a subsequence converging to $(0,0,\ldots)$ in the product topology.

(b) Show that this fails with the box topology.

The sequence $\{\vec{e}_n\}_{n=1}^{\infty}$ with $\vec{e}_n = (0,0,\ldots,0,\underset{\substack{\uparrow \\ n\text{'th component}}}{1},0,\ldots)$ has no converging subsequence in the box topology.