

Assignment 9

1. [Y] Sec. 3.3 #2.
2. [Y] Sec. 3.4 #2 (first go over the details in the proof of part (1) of Theorem 3.4).

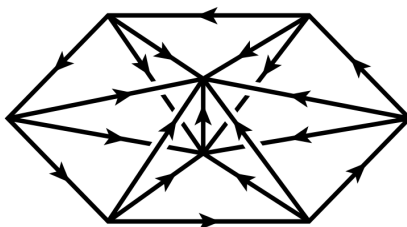
Analogous to the projective plane, the 3-dimensional real projective space $\mathbb{R}P^3$ can be obtained by identifying antipodal points of the 3-sphere S^3 , or by identifying antipodal points on the boundary S^2 of the solid 3-ball D^3 , or by equipping the set of 1-dimensional subspaces of \mathbb{R}^4 with a suitable topology.

3. This last description gives $\mathbb{R}P^3$ the structure of a 3-manifold as follows. Each point of $\mathbb{R}P^3$ has a *homogeneous coordinate* $(x_0 : x_1 : x_2 : x_3)$, i.e., the points (x_0, x_1, x_2, x_3) and $(\lambda x_0, \lambda x_1, \lambda x_2, \lambda x_3)$ of $\mathbb{R}^4 - \{(0, 0, 0, 0)\}$ are identified as the same point in $\mathbb{R}P^3$ for any nonzero real number λ .

Based on this, specify the local charts that cover $\mathbb{R}P^3$, each of which is homeomorphic to \mathbb{E}^3 , and write down the transition functions on their pairwise overlaps.

In the following, let us give two more descriptions for $\mathbb{R}P^3$, first as a *lens space* (introduced by Tietze) with the structure of a simplicial complex (see, e.g., Section 3.2 of the reference [B]), second as a quotient space obtained by a *Dehn surgery*.¹

4. Construct a 3-dimensional simplicial complex from n tetrahedra (i.e., 3-simplices) T_1, \dots, T_n by the following two steps. First arrange the tetrahedra in a cyclic pattern as in the figure, so that each T_i shares a common vertical face with its two neighbors T_{i-1} and T_{i+1} , subscripts being taken mod n . Then identify the bottom face of T_i with the top face of T_{i+1} for each i .



This simplicial complex, or its polytope (geometric realization), is an example of a *lens space*, denoted by $L(n, 1)$.

- (a) Show that $L(2, 1)$ is homeomorphic to $\mathbb{R}P^3$.
- (b) Calculate the Euler characteristic of $\mathbb{R}P^3$ by carefully enumerating the simplices of $L(2, 1)$.

¹The figures are copied from Allen Hatcher's *Algebraic topology* and John Luecke's *Dehn surgery on knots in the 3-sphere*. The descriptions below are adapted in addition from Joshua Evan Greene's *Heegaard Floer homology*.

