Assignment 12

1. [Y] Sec. 4.2 #4.

2. [Y] Sec. 4.2 #5.

3. [Y] Sec. 4.2 #6.

4. [M] Sec. 52 #7 (recall that a topological group is a group G endowed with a topology such that the group multiplication and taking inverse are continuous operations, i.e., the maps $G \times G \to G, (g_1, g_2) \mapsto g_1 g_2$ and $G \to G, g \mapsto g^{-1}$ are continuous).

5. [Y] Sec. 4.3 #3.

6. (a) Suppose that f and g are continuous maps $X \to Y$, and $H : X \times [0,1] \to Y$ is a homotopy from f to g. Fix a base point $x \in X$. Show that there exists a path $a : [0,1] \to Y$, starting at $f(x)$ and ending at $g(x)$, such that $f_*(\gamma) = [a] * g_*(\gamma) * [a]^{-1}$ for all $\gamma \in \pi_1(X, x)$.

   (b) Using part (a), prove the following. Suppose that $f : X \to Y$ and $g : Y \to X$ are continuous maps such that $g \circ f$ is homotopic to id_X and $f \circ g$ is homotopic to id_Y. Then for any base point $x \in X$, the map $f_* : \pi_1(X, x) \to \pi_1\big(Y, f(x)\big)$ is an isomorphism.

7. (a) Show that there does not exist a retraction from D^2 to S^1. (Hint: use the functoriality of the fundamental group.)

   (b) Show that any continuous map $f : D^2 \to D^2$ must have a fixed point, i.e., $x_0 \in D^2$ such that $f(x_0) = x_0$. (Hint: construct a retraction from D^2 to S^1 if there were no fixed point. The same strategy works for all D^n with π_1 replaced by higher homotopy groups π_n or homology groups H_n.)

8. [Y] Sec. 4.5 #3.

9. [Y] Sec. 4.5 #5.

10. [Y] Sec. 4.5 #10.