Assignment 10

1. Recall that a topology $\mathscr{T}$ can be specified by a *subbasis* $\mathscr{S}$, so that its basis $\mathscr{B}$ consists of all finite intersections of elements of $\mathscr{S}$, and $\mathscr{T}$ consists of all arbitrary unions of elements of $\mathscr{B}$.

   Let X and Y be topological spaces. Denote by $\mathscr{C}(X, Y)$ the set of continuous maps from X to Y. Given any $C \subset X$ compact and $U \subset Y$ open, write

   $$S(C, U) = \{f \in \mathscr{C}(X, Y) \mid f(C) \subset U\}$$

   Define the *compact–open topology* on $\mathscr{C}(X, Y)$ to be generated by the subbasis $\mathscr{S}$ consisting of sets of the form $S(C, U)$.

   (a) Check that $\mathscr{S}$ is indeed a subbasis.

   (b) Given a third space Z, define a map $\phi\colon \mathscr{C}(X \times Y, Z) \to \mathscr{C}\big(X, \mathscr{C}(Y, Z)\big)$ by sending $f : X \times Y \to Z$ to $F \colon X \to \mathscr{C}(Y, Z)$ such that each $x \in X$ maps to

   $$\begin{aligned} F(x) \colon Y &\to Z \\ y &\mapsto f(x, y) \end{aligned}$$

   Check that ϕ is well-defined, i.e., each $F(x)$ is continuous and F is also continuous. (Hint: write $F(x)$ as a composite of continuous maps. To show the continuity of F, for each fixed $x_0 \in X$, you need only consider neighborhoods of $F(x_0)$ that are in the subbasis $\mathscr{S}$. Use compactness in the definition of $\mathscr{S}$.)

   (c) With additional mild assumptions on the spaces involved, one can show that the map ϕ in part (b) is a homeomorphism. Based on this fact, explain why a homotopy between continuous maps from Y to Z is equivalent to a path in $\mathscr{C}(Y, Z)$.

2. [Y] Sec. 4.1 #2.

3. [Y] Sec. 4.1 #3.

4. [Y] Sec. 4.1 #4.