Examples of universal properties

- The pasting lemma (coproduct)

① (Cf. ④ below)

$$\begin{tikzcd}
 & X \arrow[d] \arrow[ddr, bend left] & \\
Y \arrow[r] \arrow[drr, bend right] & X \amalg Y \arrow[dr, dashed, "\exists!"] & \\
 & & Z
\end{tikzcd}$$

in the category of topological spaces (and continuous maps)

$X \amalg Y$ = disjoint union of X and Y, with

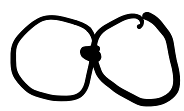
$$\mathcal{T}_{X \amalg Y} = \{ U \subset X \amalg Y \mid U \cap X \in \mathcal{T}_X \text{ and } U \cap Y \in \mathcal{T}_Y \}$$

②

$$\begin{tikzcd}
 & X \arrow[d] \arrow[ddr, bend left] & \\
Y \arrow[r] \arrow[drr, bend right] & X \vee Y \arrow[dr, dashed] & \\
 & & Z
\end{tikzcd}$$

in the category of pointed spaces (X, x_0) ← base point

(and basepoint-preserving continuous maps)

$X \vee Y$ = wedge sum of (X, x_0) and (Y, y_0), with base point $x_0 = y_0$ ("一点并"  $S^1 \vee S^1$)

$= X \amalg Y / x_0 \sim y_0$

③

$$\begin{tikzcd}
 & M \arrow[d] \arrow[ddr, bend left] & \\
N \arrow[r] \arrow[drr, bend right] & M \oplus_R N \arrow[dr, dashed] & \\
 & & L
\end{tikzcd}$$

in the category of R-modules

In particular, when $R = \mathbb{Z}$

$$\begin{tikzcd}
 & G \arrow[d] \arrow[ddr, bend left] & \\
H \arrow[r] \arrow[drr, bend right] & G \oplus H \arrow[dr, dashed] & \\
 & & K
\end{tikzcd}$$

in the category of abelian groups

(for general groups, get free product)
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when $R = k \supset \mathbb{Q}$

$$\begin{tikzcd}
 & V \arrow[d] \arrow[ddr, bend left] & \\
W \arrow[r] \arrow[drr, bend right] & V \oplus W \arrow[dr, dashed] & \\
 & & Z
\end{tikzcd}$$

in the category of k-vector spaces

④
$$\begin{tikzcd}
A \cap B \arrow[r] \arrow[d] & A \arrow[d] \arrow[ddr, bend left] & \\
B \arrow[r] \arrow[drr, bend right] & A \cup B \arrow[dr, dashed] & \\
 & & Y
\end{tikzcd}$$

in the category of topological spaces

(Note that here, in the original pasting lemma, the topology on $A \cup B$ is a little subtle. If $X = A \cup B$ is a topological space, then the diagram exists if both A and B are closed (or open) subsets of X. Compare the topology on $X \sqcup Y$ in ①.)

⑤
$$\begin{tikzcd}
A \cap B \arrow[r] \arrow[d, no head] & A \arrow[d, no head] \arrow[dr, no head] & \\
{} & {} & {}
\end{tikzcd}$$

in the category of sets

$$\begin{tikzcd}
B \arrow[r] \arrow[rr, bend right] & A \cup B \arrow[r, dashed] & Y
\end{tikzcd}$$

($A \cup B$ is the (fiber) coproduct of A and B over $A \cap B$.)

Also

$$\begin{tikzcd}
Z \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& A \cap B \arrow[r] \arrow[d] & A \arrow[d] \\
& B \arrow[r] & A \cup B
\end{tikzcd}$$

in the category of sets

($A \cap B$ is the (fiber) product of A and B over $A \cup B$.)

⑥
$$\begin{tikzcd}
R \arrow[r] \arrow[d] & A \arrow[d] \arrow[ddr, bend left] & \\
B \arrow[r] \arrow[drr, bend right] & A \otimes_R B \arrow[dr, dashed] & \\
& & C
\end{tikzcd}$$

in the category of commutative R-algebras

- Maps into a product space (product)

$$\begin{tikzcd}
Z \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times Y \arrow[r] \arrow[d] & X \\
& Y &
\end{tikzcd}$$

in the category of topological spaces

$X \times Y$ equipped with the product topology

More generally,

Y

$$\begin{tikzcd} Y \arrow[r, dashed] \arrow[rr, bend left] & \prod\limits_{\alpha\in A} X_\alpha \arrow[r, "p_\alpha"] & X_\alpha \end{tikzcd} \quad \text{for each } \alpha \in A$$

$\prod X_\alpha$ with the product topology (different from the box topology if A is infinite)

- Kernel of a group homomorphism

$$\begin{tikzcd} \ker\varphi \arrow[r] & G \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "e"'] & H \\ K \arrow[u, dashed] \arrow[ur] & & \end{tikzcd}$$

where e is the constant map to the identity of H and φ is a given homomorphism

Any homomorphism $\psi: K \to G$ such that $\varphi\circ\psi = e\circ\psi$ factors uniquely through $\ker\varphi$.

Kernel and product are both limits.
in the categorical sense

(Another example of limit is the p-adic integers $\mathbb{Z}_p$

$$\begin{tikzcd} \mathbb{Z}_p = \lim\limits_{n} \mathbb{Z}/p^n\mathbb{Z} \arrow[r] & \cdots \arrow[r] & \mathbb{Z}/p^3\mathbb{Z} \arrow[r] & \mathbb{Z}/p^2\mathbb{Z} \arrow[r] & \mathbb{Z}/p\mathbb{Z} \\ G \arrow[u, dashed] \arrow[urr] \arrow[urrr] \arrow[urrrr] & & & & \end{tikzcd}$$

in the category of abelian groups.)

- Cokernel of a group homomorphism

$$\begin{tikzcd} G \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "e"'] & H \arrow[r] \arrow[dr, bend right] & \operatorname{coker}\varphi := H/\operatorname{im}\varphi \arrow[d, dashed] \\ & & K \end{tikzcd}$$

in the category of abelian groups

In particular, if φ is the inclusion of a subgroup, $\operatorname{coker}\varphi$ is precisely the quotient group H/G: any homomorphism $H \to K$ where the subgroup $G \subset H$ maps to the identity of K must factor uniquely through H/G.

Cokernel and coproduct are both colimits.

Another example of quotient/coproduct (colimit):

$$\begin{tikzcd} p^{-1}(y) \arrow[r] \arrow[d] & X \arrow[d, "p"'] \arrow[ddr, bend left, "f"] & \\ \{y\} \arrow[r] \arrow[drr, bend right] & Y \arrow[dr, dashed] & \\ & & Z \end{tikzcd}$$

for each $y \in Y$

f is constant on each $p^{-1}(y)$

in the category of sets

in the category of topological spaces

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "f"] & \\ Y \arrow[r, dashed] & Z \end{tikzcd}$$

f is constant on each $p^{-1}(y)$

in the (sub)category of topological spaces and quotient maps

Cf. Munkres' Theorem 22.2

Cf. Munkres's Theorem 22.2.

- Tensor products of R-modules

$$\begin{tikzcd} M \times N \arrow[r] \arrow[dr, bend right, "\varphi"'] & M \otimes_R N \arrow[d, dashed] \\ & G \end{tikzcd}$$

φ is R-linear in each variable

in the category of abelian groups

Analogously,

"smash product"

$$\begin{tikzcd} X \times Y \arrow[r] \arrow[dr, bend right, "f"'] & X \wedge Y := X \times Y / X \vee Y \arrow[d, dashed] \\ & Z \end{tikzcd}$$

f is basepoint-preserving in each variable

in the category of pointed topological spaces