MA323 Topology Midterm Exam

2:00–3:50 pm

November 3, 2020

Your name: ________________ SID: ________________

1. (10 points) Show that the countable collection

$$\mathcal{B} = \{(a, b) \mid a, b \in \mathbf{Q}, a < b\}$$

is a basis that generates the standard topology on $\mathbf{R}$.

The standard topology on $\mathbb{R}$ is generated by open intervals (x, y), $x, y \in \mathbb{R}$, $x < y$. Thus it suffices to show that each (x, y) is a union of elements in $\mathcal{B}$. Indeed, since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exist sequences $\{a_n\}$ and $\{b_n\}$ in $\mathbb{Q}$ such that $a_n \to x$ from the right and $b_n \to y$ from the left with $a_n < b_n$ for each n. Therefore $(x, y) = \bigcup_{n=1}^{\infty} (a_n, b_n)$.

2. (40 points) Let X be a set. Let $\mathcal{T}_c$ be the collection of all subsets U of X such that $X - U$ is either countable or all of X.

   (a) (5 points) Show that $\mathcal{T}_c$ is a topology on X.

(i) Since ϕ is countable, $X \in \mathcal{T}_c$.
Since $X - \phi = X$, $\phi \in \mathcal{T}_c$.

(ii) Given $\{U_\alpha \mid \alpha \in A\} \subset \mathcal{T}_c$, $X - \bigcup_{\alpha \in A} U_\alpha = \bigcap_{\alpha \in A}(X - U_\alpha)$.
If $X - U_\alpha$ is countable for some α, then $\bigcap_{\alpha \in A}(X - U_\alpha)$ is countable. If $X - U_\alpha = X$ for all α, then $\bigcap_{\alpha \in A}(X - U_\alpha) = X$.
Therefore $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}_c$.

(iii) Given $\{U_i \mid 1 \leq i \leq n\} \subset \mathcal{T}_c$, $X - \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n}(X - U_i)$.
If $X - U_i = X$ for some i, then $\bigcup_{i=1}^{n}(X - U_i) = X$. Otherwise, as a finite union of countable sets, $\bigcup_{i=1}^{n}(X - U_i)$ is countable.
Therefore $\bigcap_{i=1}^{n} U_i \in \mathcal{T}_c$.

For the rest of the question, X denotes the topological space $(X, \mathcal{T}_c)$.

   (b) (5 points) Suppose that A is an uncountable subset of X. Show that $\overline{A} = X$.

It suffices to show that given any $U \in \mathcal{T}_c$, $U \cap A \neq \phi$. Since $X - U$ is countable, $A \not\subset X - U$. Thus $U \cap A \neq \phi$.

(c) (5 points) Suppose in addition that A is proper and let $x \in X - A$. Show that x is a limit point of A.

We need to show that given any neighborhood U of x, $(U - \{x\}) \cap A \neq \phi$. It suffices to show that $X - (U - \{x\})$ is countable. Note that $X - (U - \{x\}) = (X - U) \cup \{x\}$ and $X - U$ is countable.

Alternatively, by part (b), since $X = \bar{A} = A \cup A'$, we have $X - A \subset A'$.

(d) (10 points) Given x from part (c), show that every neighborhood of x contains infinitely many points in A. (Hint: first show that X satisfies the T_1 axiom.)

Continuing the proof of part (c), let $a_1 \in (U - \{x\}) \cap A$. Then $U - \{a_1\}$ is a neighborhood of x.

Inductively we obtain a sequence $\{a_i\}_{i=1}^{\infty}$ of distinct points in A that are contained in U.

(Did not use the hint. Cf. Theorem 17.9 of the textbook.)

Alternatively, suppose that $\{x_n\}$ in A converges to x. Since $x \notin A$, $X - \{x_n, n \in \mathbb{Z}_+\}$ is a neighborhood of x, so it contains all but finitely many points x_n, a contradiction.

(e) (10 points) In spite of part (d), show that any sequence of points in A does not converge to x. (Hint: first show that every convergent sequence in A *stabilizes*, i.e., is eventually constant.)

Since $x \notin A$, we need only show the statement of the hint. Let $\{a_n\}$ be a convergent sequence in A, with a its limit. Let $K = \{a_n \mid a_n \neq a\}$. Then $X - K$ is a neighborhood of a and so it contains all but finitely many points a_n. Thus K is finite.

(f) (5 points) Show that X is not metrizable. (Hint: show that in a metric space, any point in the closure of a subset is the limit of a sequence in the subset.)

By part (b), $x \in \overline{A}$. In view of part (e), it suffices to prove the claim in the hint. Let Y be a metric space, $D \subset Y$, and $y \in \overline{D}$. For each $n \in \mathbb{Z}_+$, let $d_n \in D \cap B(y, \frac{1}{n})$.

Then the sequence $\{d_n\}$ converges to y. In fact, given any neighborhood V of y, there exists $B(y, r) \subset V$ for some $r > 0$. Choose $N \in \mathbb{Z}_+$ such that $\frac{1}{N} < r$. Then $d_n \in V$ for all $n \geq N$.

Recall that the condition for a surjection $f: X \to Y$ to be a quotient map is that $V \subset Y$ is open if and only if $f^{-1}(V)$ is open in X.

3. (10 points)

(a) (5 points) Is an open map necessarily a quotient map? Give a proof or a counterexample.

No. A continuous and surjective open map is a quotient map.

The identity map $(X, \mathcal{T}_{trivial}) \to (X, \mathcal{T}_{discrete})$ is open but not continuous.

The inclusion of an open subspace is open and continuous but not surjective.

(b) (5 points) Is a quotient map necessarily open? Give a proof or a counterexample.

No. We want to construct a counterexample where $U \subset X$ is open but $f^{-1}(f(U))$ is not open.

Consider $X = \mathbb{R}$ with the standard topology. Define an equivalence relation $\sim$ on X by declaring $x \sim y$ if and only if $x = y$ or $x, y \in \mathbb{Z}$. Endow the set $X/\sim$ of equivalence classes with the quotient topology. Then $X \to X/\sim$ is a quotient map. Let U be an open set containing an integer.

(Also see Munkres §22 #3.)

4. (20 points) Let X and Y be topological spaces and $f : X \to Y$ be a function. Define the *graph* of f to be

$$\Gamma = \{(x, y) \in X \times Y \mid y = f(x)\}$$

Consider the following statement: if Γ is a closed subset of $X \times Y$, then f is continuous.

(a) (10 points) Prove this statement or provide a counterexample.

False. Consider $f : \mathbb{R} \to \mathbb{R}$

$$x \mapsto \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

(True if Y is compact. Appeared later in Homework #8.)

(b) (10 points) Do the same for the converse.

False. Consider the identity map

$id : (X, \mathcal{T}_{discrete}) \to (X, \mathcal{T}_{trivial})$.

(True if Y is Hausdorff.)

5. (20 points)

(a) (10 points) Prove that the circle and square

$$X := \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 = 1\} \quad Y := \{(x, y) \in \mathbf{R}^2 \mid |x| + |y| = 1\}$$

are homeomorphic.

Define $f : Y \to X$

$$(x, y) \mapsto \frac{1}{\sqrt{x^2+y^2}}(x, y)$$

Check....

(b) (10 points) Prove that the open and closed unit balls

$$\{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 < 1\} \qquad \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 \leq 1\}$$

are not homeomorphic.

For example, can argue by comparing punctured neighborhoods of a boundary point, whether a loop can always continuously deforms to a constant loop.