MA323 Midterm Exam, Fall 2019

Name: ______________________

Instructions: Calculators, course notes and textbooks are **NOT** allowed on the worksheet. All numerical answers **MUST** be exact; e.g., you should write π instead of 3.14..., $\sqrt{2}$ instead of 1.414..., and $\frac{1}{3}$ instead of 0.3333... Explain your reasoning using complete sentences and correct grammar, spelling, and punctuation. Feel free to write in either English or Chinese.
Show ALL of your work!

Question 1 (21 points). True or false? You need not justify your answers.

F (a) Let X be a topological space and A be a closed subspace of X. If X is normal, then any continuous map of A into $\mathbb{R}$ may be extended to a continuous map of all of X into $\mathbb{R}$.

T (b) Recall that S_Ω denotes the minimal uncountable well-ordered set. The set $\bar{S}_\Omega = S_\Omega \cup \{\Omega\}$ with the order topology is Hausdorff.

T (c) The space $\bar{S}_\Omega$ is the one-point compactification of S_Ω.

F (d) Let $X = \mathbb{R}$ with the countable complement topology and $A \subset X$. Then given any $a \in \overline{A}$, there is a sequence in A that converges to a.

T (e) With the above notation, if a sequence in A converges to a, then $a \in \overline{A}$.

T (f) The 2-manifold $\mathbb{R}P^2$ is imbeddable into $\mathbb{R}^4$ but not $\mathbb{R}^3$.

F (g) Each connected component of a space is both ~~open~~ and closed.

Question 2 (20 points). Let X be a topological space and $\{X_\alpha\}_{\alpha\in A}$ be an indexed family of topological spaces.

(a) Let (x_n) be a sequence of points of X. State the definition of (x_n) converging to x in X.

We say that (x_n) converges to x, if given any neighborhood U of x, there exists a positive integer N_U such that $x_n \in U$ for every $n > N_U$.

(b) State the definition of the product topology on the set $\prod_{\alpha\in A} X_\alpha$.

Each open set of the product topology is a union (including the empty set) of sets of the form $\prod_{\alpha\in A} U_\alpha$ where each U_α is open in X_α and all but finitely many of them equal X_α.

(c) Suppose that $X = \prod_{\alpha\in A} X_\alpha$ with the product topology. Show that if $x_n \to x$ in X, then $\pi_\alpha(x_n) \to \pi_\alpha(x)$ for each α, where $\pi_\alpha : X \to X_\alpha$ is the projection map.

Since each π_α is continuous with respect to the product topology, given any neighborhood V of $\pi_\alpha(x)$, there exists a neighborhood U_V of x such that $\pi_\alpha(U_V) \subset V$. Since $x_n \to x$, there exists a positive integer N_V such that $x_n \in U_V$ for every $n > N_V$. Thus $\pi_\alpha(x_n) \in V$ for every $n > N_V$. Therefore $\pi_\alpha(x_n) \longrightarrow \pi_\alpha(x)$.

(d) Is the converse of the statement in part (c) true? Give a proof or a counterexample.

Yes. Given any neighborhood U of x, there exists a neighborhood of x contained in U that is of the form $\prod_{\alpha\in A} U_\alpha$ where each U_α is open in X_α and all but finitely many of them equal X_α. For each $U_\alpha \neq X_\alpha$, since $\pi_\alpha(x_n) \to \pi_\alpha(x)$, there exists a positive integer N_α such that $\pi_\alpha(x_n) \in U_\alpha$ for every $n > N_\alpha$. Let N be the largest among the integers N_α. Then $x_n \in \prod_{\alpha\in A} U_\alpha \subset U$ for every $n > N$. Therefore $x_n \to x$.

Question 3 (25 points). Let X be a topological space.

(a) State the definition of X being regular.

A space X is said to be regular if the following conditions hold:

- Given any distinct points $x, y \in X$, there exists an open set containing x but not containing y;
- Given any point $x \in X$ and any closed set A not containg x, there exist open sets U containing x and V containing A such that $U \cap V = \phi$.

(b) State the definition of X being normal.

A space X is said to be normal if the following conditions hold:

- Given any distinct points $x, y \in X$, there exists an open set containing x but not containing y;
- Given any disjoint closed sets A and B, there exist open sets U containing A and V containing B such that $U \cap V = \phi$.

(c) Show that if X is compact and Hausdorff, then it is regular.

Since X is Hausdorff, the first condition in the definition is satisfied. Let $x \in X$ and A be a closed subset of X not containing x. Since X is compact, so is A. Given any $a \in A$, since X is Hausdorff, there exist open sets U_a and V_a such that $x \in U_a$, $a \in V_a$, and $U_a \cap V_a = \phi$. Then $\{V_a \mid a \in A\}$ is an open covering of A and thus has a finite subcovering $\{V_{a_i} \mid a_i \in A, i = 1, \cdots, n\}$. Let $U = \bigcap_{i=1}^{n} U_{a_i}$ and $V = \bigcup_{i=1}^{n} V_{a_i}$. Then U and V are open sets such that $x \in U$, $A \subset V$, and $U \cap V = \phi$.

(d) Show that if X is compact and Hausdorff, then it is normal.

The proof is analogous to that of part (c). Replace x with a closed set B disjoint from A. Then use (c) to construct $U_a \supset B$.

(e) Is the converse of the statement in part (d) true? Give a proof or a counterexample.

No. Let $X = \mathbb{R}$ with the standard euclidean topology. Then X is not compact. Since X is a metric space, X is normal.

Question 4 (20 points). Let $f : X \to Y$, where Y is Hausdorff. Write $G_f = \{(x, f(x)) | x \in X\}$.

(a) Show that if f is continuous, then G_f is closed in $X \times Y$.

Suppose that $(x, y) \in X \times Y - G_f$, i.e., $y \neq f(x)$. Since Y is Hausdorff, there are disjoint neighborhoods V, W of y and $f(x)$ respectively. Since f is continuous, there is a neighborhood U of x such that $f(U) \subset W$. Thus $U \times V$ is a neighborhood of (x, y) disjoint with G_f. This implies that $X \times Y - G_f$ is open and so G_f is closed.

(b) Show that if Y is compact, then the projection $\pi_X : X \times Y \to X$ is a closed map.

Let $C \subset X \times Y$ be closed. Suppose $x \notin \pi_X(C)$. Then for each $y \in Y$, $(x, y) \notin C$, and thus there are neighborhoods U_y of x and V_y of y such that $U_y \times V_y \subset X \times Y - C$. Since Y is compact, its open covering $\{V_y | y \in Y\}$ has a finite subcovering $\{V_{y_i} | y_i \in Y, i = 1, \cdots, n\}$. Then $\bigcap_{i=1}^{n} U_{y_i}$ is a neighborhood of x disjoint with $\pi_X(C)$. Therefore $\pi_X(C)$ is closed and so π_X is a closed map.

(c) Show that if Y is compact and G_f is closed, then f is continuous.

Let $D \subset Y$ be closed. Then $f^{-1}(D) = \pi_X(G_f \cap X \times D)$. Since G_f is closed, so is $G_f \cap X \times D$. By part (b), since Y is compact, $\pi_X(G_f \cap X \times D)$ is closed. Therefore f is continuous.

(d) Without Y being compact, is the statement in part (c) still true? Give a proof or a counterexample.

No. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

Question 5 (14 points). Let $A \subset \mathbb{R}^2$.

(a) Show that if A is connected and open, then it is path connected. (Hint: Show that given $a \in A$, the set of points that can be joined to a by a path in A is both open and closed in A.)

Given $a \in A$, let B be the set of points that can be joined to a by a path in A. Then $a \in B$ so $B \neq \phi$. To show that B is open, let $b \in B$, α be a path from a to b, and U be an open disk such that $b \in U \subset A$. Since U is path connected, for any $x \in U$ there is a path β in A from b to x and thus $\alpha * \beta$ joins a and x so $x \in B$. Therefore B is open. To show B is closed, suppose $c \in A-B$. Let V be an open disk such that $c \in V \subset A$. If there were $y \in V \cap B$, then a path from a to y followed by a path from y to c would contradict $c \notin B$. Thus $V \subset A-B$ so $A-B$ is open. Therefore B is closed. Since A is connected, $B=A$, so A is path connected.

(b) Show that if A is countable, then $\mathbb{R}^2 - A$ is path connected. (Hint: How many lines are there passing through a given point of $\mathbb{R}^2$?)

Let $x, y \in \mathbb{R}^2 - A$ with $x \neq y$. Since there are uncountably many lines passing through x, one of them, say L, contains no point of A. If L passes through y, then we are done. If not, let K be a line through y that contains no point of A and is not parallel to L. Let z be the intersection of K and L. Then the line segment from x to z and that from z to y give a path joining x and y.