MA301, Fall 2017

Midterm 2

Due in-class on **Wednesday, December 13**

The ONLY source you may consult is the textbook. All questions have equal value.

In the following, all functions will be complex valued. We shall write xy instead of $x \cdot y$ if x, y are elements of $\mathbb{R}^n$.

1. Let f be a function on $\mathbb{R}^n$. We say that f *tends to* 0 *rapidly at infinity* if for each positive integer d the function

$$x \mapsto |x|^d f(x)$$

   is bounded for $|x|$ sufficiently large. Write $\mathscr{S}$ for the set of functions on $\mathbb{R}^n$ which are infinitely differentiable (i.e., partial derivatives of all orders exist and are continuous), and which tend to 0 rapidly at infinity, as well as their partial derivatives of all orders.

   (i) Verify that $e^{-x^2} \in \mathscr{S}$ (remember that $x^2 = x \cdot x$). (Another example will appear in Lemma 14.4 of the textbook.)

   (ii) Show that any function in $\mathscr{S}$ is integrable.

2. Define the *Fourier transform* of a function $f \in \mathscr{S}$ by

$$\widehat{f}(y) = \int_{\mathbb{R}^n} f(x)\, e^{-2\pi i x y}\, dx$$

   (i) Let D_j be the partial derivative with respect to the j'th variable. For each n-tuple $p = (p_1, \ldots, p_n)$ of nonnegative integers, we write $D^p = D_1^{p_1} \cdots D_n^{p_n}$. Similarly, given $f \in \mathscr{S}$, let $M_j f$ be the function such that $(M_j f)(x) = x_j f(x)$, i.e., multiplication by the j'th variable, and let $(M^p f)(x) = x_1^{p_1} \cdots x_n^{p_n} f(x)$. Show that

$$D^p \widehat{f} = (-2\pi i)^{|p|}\, \widehat{M^p f} \qquad \text{and} \qquad M^p \widehat{f} = (2\pi i)^{-|p|}\, \widehat{D^p f}$$

   where $|p| = p_1 + \cdots + p_n$. Hint: for the first identity, inductively, differentiate across the integral sign (justification needed); for the second, integrate by parts.

   (ii) Show that the Fourier transformation $f \mapsto \widehat{f}$ is a linear map of the complex vector space $\mathscr{S}$ into itself.

3. A function g on $\mathbb{R}^n$ is called *periodic* if $g(x+m) = g(x)$ for all $m \in \mathbb{Z}^n$. Given a periodic C^∞ function g, we define its *k'th Fourier coefficient* by

$$c_k = \int_{T^n} g(x)\, e^{-2\pi i k x}\, dx \qquad\qquad k \in \mathbb{Z}^n$$

   where $T^n = \mathbb{R}^n/\mathbb{Z}^n$ is the n-torus, and the integral over T^n is by definition the n-fold integral with the variables $(x_1, \ldots, x_n)$ ranging from 0 to 1.

(i) Show that the *Fourier series*

$$\sum_{k\in\mathbb{Z}^n} c_k\, e^{2\pi i k x}$$

converges to g uniformly. (Cf. Corollary 7.17 of Browder.)

(ii) Let $f \in \mathscr{S}$. Show that

$$\sum_{m\in\mathbb{Z}^n} f(m) = \sum_{m\in\mathbb{Z}^n} \widehat{f}(m)$$

Hint: consider $g(x) = \sum_{m\in\mathbb{Z}^n} f(x+m)$.