MA301, Fall 2017

Midterm 1

Due in-class on **Wednesday, November 1**

The ONLY source you may consult is the textbook. All questions have equal value.

1. Let X, Y be compact metric spaces. If f, g are continuous real-valued functions on X and Y respectively, we denote by $f \otimes g$ the function such that

$$(f \otimes g)(x, y) = f(x)\, g(y)$$

   Show that every continuous function on $X \times Y$ can be uniformly approximated by sums $\sum_{i=1}^{n} f_i \otimes g_i$ where f_i is continuous on X and g_i is continuous on Y.

2. Let $\mathscr{F} \subset C(X)$ be equicontinuous. Let B be the set of points $x \in X$ such that $\mathscr{F}(x) := \{f(x) \,|\, f \in \mathscr{F}\}$ is bounded. Prove that B is open and closed in X. If X is compact and connected, and if for some point $a \in X$ the set $\mathscr{F}(a)$ is bounded, show that $\mathscr{F}$ is relatively compact in $C(X)$.

3. (i) A collection $\mathscr{S}$ of subsets of X is said to be *monotone* if, whenever $\{A_n\}$ is an increasing (resp. decreasing) sequence of subsets in $\mathscr{S}$, then

$$\bigcup_{n=1}^{\infty} A_n \quad (\text{resp. } \bigcap_{n=1}^{\infty} A_n)$$

   also lies in $\mathscr{S}$. Let $\mathscr{A}$ be an algebra of subsets of X. Show that there exists a smallest monotone collection of subsets of X containing $\mathscr{A}$. Denote it by $\mathscr{M}$. Show that $\mathscr{M}$ is a σ-algebra, and is thus the smallest σ-algebra containing $\mathscr{A}$.

   (ii) Let $\mu\colon \mathscr{A} \to [0, \infty)$ be countably additive. Show that an extension of μ to a measure on $\mathscr{M}$, if it exists, must be unique. Give an example of $(\mathscr{A}, \mathscr{M}, \mu)$.

4. Let X be a complete metric space and μ a measure on X. Suppose that for any $\epsilon > 0$ there exists an open set D dense in X such that $\mu(D) < \epsilon$. Show that there is a dense subset of X whose measure is zero. Give an example of (X, μ, D).

5. (i) Let l^1 be the set of all sequences $\alpha = \{a_n\}$ in $\mathbb{R}$ such that $\sum |a_n|$ converges. Define

$$|\alpha| = \sum |a_n|$$

   Show that this makes l^1 into a complete metric space.

   (ii) Let $\beta = \{b_n\}$ be a fixed sequence in l^1. Show that the set of all $\alpha \in l^1$ such that $|a_n| \leq |b_n|$ is compact. Show that the unit closed ball in l^1 is not compact.