MA301, Fall 2017

Final Exam

This test is due on or by **Friday, January 19** by 1 pm. Return the exam to my office in Huiyuan 3-419 by the same time. No collaboration is allowed. The only source you may consult is the textbook. All questions have equal value.

1. Let $\{f_n\}$ be a sequence in $L^2(X,\mu)$ such that $\|f_n\|_2 \to 0$ as $n \to \infty$. Show that

$$\lim_{n\to\infty} \int_X |f_n(x)| \log\big(1 + |f_n(x)|\big)\, d\mu(x) = 0$$

2. Let $1 \le p < \infty$ and let $f \in L^p(\mathbb{R}, m)$. Given $a \in \mathbb{R}$, let T_a be the translation by a, that is, $T_a f(x) = f(x-a)$. Show that $T_a f$ converges to f in L_p as $a \to 0$. Is this conclusion still true if $p = \infty$?

The remaining three questions form a sequence.

3. Let s be an integer. On the integers $\mathbb{Z}$, define

$$\mu_s(n) = (1+n^2)^s$$

Then μ_s extends to a measure on $\mathbb{Z}$ (we have just defined its values on one-point sets).

   (i) Write H_s for the space of complex-valued functions on $\mathbb{Z}$, written as sequences $\{a_n\}_{n\in\mathbb{Z}}$ of complex numbers, such that

$$\sum (1+n^2)^s |a_n|^2$$

   converges. If $f = \{a_n\}$ and $g = \{b_n\}$, define the inner product in H_s to be

$$\langle f, g\rangle = \sum a_n \bar{b}_n (1+n^2)^s$$

   Show that $H_s = L^2(\mathbb{Z}, \mu_s)$, and in particular H_s is complete for the norm associated with this inner product.

   (ii) Show that the finite sequences $f = \{a_n\}$ with $a_n = 0$ for all but a finite number of n form a dense subspace of H_s.

4. To each $f \in C^\infty(T)$, where $T = \mathbb{R}/\mathbb{Z}$ is the circle, associate (again) the Fourier series

$$f(x) = \sum a_n e^{2\pi i n x} \qquad \text{with} \quad a_n = \int_0^1 f(t) e^{-2\pi i n t}\, dt$$

   (i) Integrating by parts, show that the coefficients satisfy the inequality

$$|a_n| \ll \frac{1}{|n|^k}$$

   for each positive integer k. The symbol $\ll$ means that the left-hand side is less than some constant times the right-hand side for $|n| \to \infty$.

(ii) Show that $C^\infty(T) \subset L^2(\mathbb{Z}, \mu_s)$ for all $s \in \mathbb{Z}$, and that $C^\infty(T)$ is a dense subspace.

5. Let $r < s$. Show that the unit ball in H_s is relatively compact in H_r, in other words, it is totally bounded. (This should be compared to the last question in #5 of Midterm 1.)