Presentation Topics 3

1. Prove that if $n \in \mathbb{N}$, then
$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2^n - 1} + \frac{1}{2^n} \geq 1 + \frac{n}{2}.$$
What does this mean if $n \to \infty$?

2. Define a sequence of numbers $a_0, a_1, a_2, \ldots$ by setting $a_{n+1} = 2a_n - a_n^2$ for all $n \geq 0$. Show that $a_n = 1 - (1 - a_0)^{2^n}$ for all $n = 0, 1, 2, \ldots$. What happens for different choices of a_0?

3. Prove that the following are equivalent for a nonempty set A:
   - A is countable.
   - There exist an injection $f \colon A \to \mathbb{N}$.
   - There exists a surjection $g \colon \mathbb{N} \to A$.

4. Show that, $|A| < |\mathcal{P}(A)|$, for any set A.

5. Prove that if $|A| \leq |B|$, then $|\mathcal{P}(A)| \leq |\mathcal{P}(B)|$. Prove that if $|A| = |B|$, then $|\mathcal{P}(A)| = |\mathcal{P}(B)|$.

6. Let I be a countable set and $\{A_i\}_{i \in I}$ an I-indexed collection of sets. Prove that if all A_i are countably infinite, then so is $\bigcup_{i \in I} A_i$. Is the converse true?

7. Let $A, B \subseteq U$ with A countably infinite and B uncountable. Prove that $B \setminus A$ is uncountable. Give examples to show that if A, B are both uncountable, then $B \setminus A$ can be any of: empty, finite and nonempty, countably infinite, or uncountable.

8. Let A be an infinite set. Prove that $A, \mathcal{P}(A), \mathcal{P}(\mathcal{P}(A)), \ldots$ is an infinite list of infinite sets, all of which have different cardinalities. Find a set whose cardinality is greater than all of the sets on the list.

9. Let I be a set. Recall that $\prod_{i\in I}\mathbb{Z}$ is the set of all functions $I \to \mathbb{Z}$. Define $\bigoplus_{i\in I}\mathbb{Z} \subseteq \prod_{i\in I}\mathbb{Z}$ to be the subset consisting of those $f : I \to \mathbb{Z}$ such that $f(i) = 0$ for all but finitely many $i \in I$. Prove that $\prod_{i\in I}\mathbb{Z} = \bigoplus_{i\in I}\mathbb{Z}$ if and only if I is finite. Prove that if I is countable, then so is $\bigoplus_{i\in I}\mathbb{Z}$.

10. Let $\{A_i\}_{i\in\mathbb{Z}^+}$ be a $\mathbb{Z}^+$-indexed collection of sets. What are the most general conditions on the sets A_i for which $\prod_{i\in\mathbb{Z}^+} A_i$ is countable?

11. Let $\mathcal{C} \subset \mathcal{P}(\mathbb{N})$ be the collection of those subsets of $\mathbb{N}$ that have at most 10 elements. Prove that $\mathcal{C}$ is countable. Let $\mathcal{D} \subset \mathcal{P}(\mathbb{N})$ be the set of **finite** subsets of $\mathbb{N}$. Prove that $\mathcal{D}$ is countable.

12. Show that there are uncountably many **surjective** functions $f : \mathbb{N} \to \mathbb{N}$.

13. Let $\ell_1, \ell_2, \ldots$ be countably many lines in the plane $\mathbb{R}^2$. Show that there is a point in $\mathbb{R}^2$ that does not lie on any of the lines ℓ_i. (Hint: Show that there is a line whose slope is different from the slopes of all the ℓ_is).