Presentation Topics 2

1. Let A, B be sets and let $f\colon A \to B$ be an injective function. If $X, Y \subseteq A$, show that $f(X) \subseteq f(Y)$ implies that $X \subseteq Y$. Give an example to show that the assumption of injectivity is necessary.

2. Let A, B be sets and let $f\colon A \to B$ be an injective function. If $X \subseteq A$, show that $f^{-1}(f(X)) = X$. Give an example to show that the assumption of injectivity is necessary.

3. Let A, B, C be sets. If $f\colon A \to B$, $g\colon B \to C$, and $h\colon C \to D$ are functions, show that $h(gf) = hg(f)$. What is the analogous statement for more functions?

4. Let A, B, C be sets and let $f\colon A \to B$ and $g\colon B \to C$ be functions. Show that if gf is surjective, then so is g. Give an example to show that the assumption of surjectivity is necessary.

5. Let A, B, C be sets and let $f\colon A \to B$ and $g\colon B \to C$ be functions. Show that if gf is injective, then so is f. Give an example to show that the assumption of injectivitiy is necessary.

6. Let A, B, C be sets and let $f\colon A \to B$ and $g\colon B \to C$ be functions. Show that if f and g are invertible, show that so is gf and $(gf)^{-1} = f^{-1}g^{-1}$. Give an example to show that the assumption of invertibility is necessary.

7. Define a relation R on the set of real numbers $\mathbb{R}$ by aRb if $a - b$ is an integer. Show that this is an equivalence relation and give a description of the equivalence classes. What happens if you define the same relation on the integers $\mathbb{Z}$ instead of on $\mathbb{R}$?

8. Consider the subset A of $\mathbb{Z} \times \mathbb{Z}$ defined as

$$A = \{(a, b)\colon a, b \in \mathbb{Z}, b \neq 0\}$$

and define a relation R on A by $(a,b)R(c,d)$ if $ad = bc$. Show that R is an equivalence relation and describe the equivalence classes of (a,b) for any $(a,b) \in A$. What happens of A is defined without the restriction on b?

9. If R and S are relations on a set A, prove or give counterexamples for the following:

   - If R and S are equivalence relations, then $R \cap S$ is an equivalence relation.
   - If R and S are equivalence relations, then $R \cup S$ is an equivalence relation.

10. Let R be the relation on $\mathbb{Z}$ defined by aRb when b is a multiple of a. Is R a partial ordering on $\mathbb{Z}$? A linear ordering? What about if $\mathbb{Z}$ were replaced by $\mathbb{N} = \{1, 2, \dots\}$?

11. Let $X = \mathcal{P}(\mathbb{N}) \setminus \{\emptyset\}$. For $R, S \in X$, define $R \sim S$ if and only if $\min R = \min S$. Show that $\sim$ is an equivalence relation on X and describe the equivalence classes. Why is $\emptyset$ excluded from X?

12. Assume that A is a set that is linearly ordered by a relation $\leq$. Define

$$\max(a,b) = \begin{cases} a & \text{if } b < a \\ b & \text{if } a \leq b. \end{cases}$$

   Show that max is a binary operation on A. Is it associative? Is it commutative?

13. By writing out the answers for the first few, guess a formula for the sum of the first n odd integers: $1+3+5+\cdots+\cdots(2n-1)$. Then prove that your formula is correct.

14. Explain Theorem 5.2.2 from the textbook. Give a proof, and also give an example (different from Example 3) of a statement that is most naturally proved using Theorem 5.2.2 rather than the usual form of induction (Theorem 5.2.1).

15. Prove that

$$1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

   for all $n \in \mathbb{N}$. Figure out a similar statement for sums of cubes.

16. Show that for all $n \in \mathbb{N}$,

$$2(\sqrt{n+1} - 1) < 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n}.$$

17. Use induction to prove the following: if x is a real number and $1 + x > 0$, then $(1+x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$.