Math 300 MIDTERM SOLUTIONS

1. (10 points) For each of the following statements,

(i) rewrite the statement without words, using symbols such as $\forall$ and $\exists$, and

(ii) negate the statement.

(a) For all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $y^2 = x$.

Rewrite:
$\forall x \in \mathbb{R},\ \exists y \in \mathbb{R}\ \ni y^2 = x.$
Or better,
$\forall x \in \mathbb{R},\ \big(\exists y_x \in \mathbb{R}\ \ni y_x^2 = x\big).$

Negate:
$\exists x_0 \in \mathbb{R}\ \ni \big(\forall y \in \mathbb{R},\ y^2 \neq x_0\big).$
There exists $x \in \mathbb{R}$ such that $x \neq y^2$ for all $y \in \mathbb{R}$.

(b) There exists $y \in \mathbb{R}$ such that $y^2 = x$ for all $x \in \mathbb{R}$.

Rewrite:
$\exists y \in \mathbb{R}\ \ni\ y^2 = x,\ \forall x \in \mathbb{R}.$
Or better,
$\exists y_0 \in \mathbb{R}\ \ni \big(\forall x \in \mathbb{R},\ y_0^2 = x\big).$

Negate:
$\forall y \in \mathbb{R},\ \big(\exists x_y \in \mathbb{R}\ \ni y^2 \neq x_y\big).$
For all $y \in \mathbb{R}$, there exists $x \in \mathbb{R}$ such that $x \neq y^2$.

2. (15 points) Consider the following implication (assume that x and y are integers).

If xy is odd then x is odd and y is odd.

(a) Write the converse of the implication.

If x and y are both odd, then xy is odd.

(b) Write the negation of the implication.

There exist x and y such that xy is odd, but either x or y is even.

$\neg(P \Rightarrow Q) \iff P \wedge \neg Q$

(Note that there is an implicit universal quantifier in the original statement.)

(c) Write the contrapositive of the implication.

If x or y is even, then xy is even.

3. (9 points) Let $A = \{a, b, c, d\}$ and $B = \{b, c, e, f, g\}$. Write down an expression for each of the following sets in terms of the sets A and B using set operations (union, intersection, complement, etc.).

(a) $\{a, b, c, d, e, f, g\}$

$A \cup B$

(b) $\{e, f, g\}$

$B - A$

(c) $\{a, d, e, f, g\}$

$(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$

4. (24 points) Let $\mathcal{P}(X)$ denote the power set of a set X.

(a) Let $A = \{1, 2, 3, 4, 5, 6\}$. For each of the following statements, write whether it is true or false.

- $\{3, 4\} \in \mathcal{P}(A)$

  True

- $\{\{3, 4\}\} \in \mathcal{P}(A)$

  False

- $\{\{3, 4\}\} \subseteq \mathcal{P}(A)$

  True

- $\{\{3\}, \{4\}\} \subseteq \mathcal{P}(A)$

  True

(b)

- What are the elements of $\mathcal{P}(\emptyset)$?

  $\emptyset$

- What are the elements of $\mathcal{P}(\mathcal{P}(\emptyset))$?

  $\emptyset, \quad \{\emptyset\}$

- List all the subsets of $\mathcal{P}(\mathcal{P}(\emptyset))$.

  $\emptyset, \quad \{\emptyset\}, \quad \{\{\emptyset\}\}, \quad \{\emptyset, \{\emptyset\}\}$

5. (18 points) The following statements are both false. Prove this by giving a counterexample for each.

(a) Let U be a universal set. If A and B are sets, then $\overline{A} \cup \overline{B} = \overline{A \cup B}$.

Set $U = \{1, 2\}$, $A = \{1\}$, and $B = \{2\}$. Then $\overline{A} = \{2\}$ and $\overline{B} = \{1\}$ so that $\overline{A} \cup \overline{B} = \{2, 1\}$. On the other hand, $A \cup B = \{1, 2\}$ and so $\overline{A \cup B} = \emptyset$. Clearly $\{2, 1\} \neq \emptyset$.

(b) Let $f\colon X \to Y$ be a function. If A and B are subsets of X, then

$$f(A \cap B) = f(A) \cap f(B).$$

Set $X = \{1, 2\}$ and $Y = \{3\}$. Define $f(1) = f(2) = 3$. Now let $A = \{1\}$ and $B = \{2\}$. Then $A \cap B = \emptyset$ so that $f(A \cap B) = \emptyset$. However, by definition of f, $f(A) = f(B) = \{3\}$, and so $f(A) \cap f(B) = \{3\}$, which is not empty.

6. (12 points) Prove that if A and B are sets, then

$$A \subseteq B \iff A \cap \overline{B} = \emptyset.$$

Proof "$\Longrightarrow$": Suppose $A \cap \overline{B} \neq \emptyset$. Let $x \in A \cap \overline{B}$. Then $x \in A$ and $x \notin B$. However, since $A \subseteq B$ and $x \in A$, we have $x \in B$, a contradiction. Hence $A \cap \overline{B} = \emptyset$.

"$\Longleftarrow$": Let $x \in A$. Since $A \cap \overline{B} = \emptyset$, we know $x \notin \overline{B}$, that is, $x \in B$. Thus $A \subseteq B$. $\square$

7. (12 points) Let $f\colon X \to Y$ be a function and $B \subseteq Y$. Prove that $f^{-1}(\overline{B}) = \overline{f^{-1}(B)}$.

Proof To show $f^{-1}(\overline{B}) \subseteq \overline{f^{-1}(B)}$, let $x \in f^{-1}(\overline{B})$. Then $f(x) \in \overline{B}$. In other words, $f(x) \notin B$, and thus $x \notin f^{-1}(B)$. Therefore $x \in \overline{f^{-1}(B)}$.

On the other hand, given $y \in \overline{f^{-1}(B)}$, we have $y \notin f^{-1}(B)$, which implies that $f(y) \notin B$. Thus $f(y) \in \overline{B}$, and so $y \in f^{-1}(\overline{B})$. Hence $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$. $\square$