1. The binomial coefficients $\binom{n}{k}$ (pronounced "n choose k") are defined for nonnegative integers n and k by the formulas

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \qquad k = 0, 1, \ldots, n$$

where $0! = 1$ and $n! = n(n-1)!$ for $n = 1, 2, \ldots$. We put $\binom{n}{k} = 0$ for $k > n$.

   (a) Show that

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$$

   for $k = 1, 2, \ldots, n$.

   (b) Show that

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

   for any nonnegative integer n, and deduce

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

   for any a, b.

   (c) Find

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots$$

   and

$$\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$$

2. Prove by induction that for any positive integers a, b, n,

$$\binom{a}{0}\binom{b}{n} + \binom{a}{1}\binom{b}{n-1} + \cdots + \binom{a}{n}\binom{b}{0} = \binom{a+b}{n}$$

3. Prove, using induction or otherwise, that

$$\sum_{k=1}^{n}(2k-1) = n^2, \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

4. Which is larger, $99^{50} + 100^{50}$ or 101^{50}? Try to answer without using a calculator or a computer.