Math 240 Quiz 3 (1.7-1.9)

NetID: ____________ **Class time:** ____________

Instructions: Calculators, course notes and textbooks are **NOT** allowed on the quiz. All numerical answers **MUST** be exact; e.g., you should write π instead of 3.14..., $\sqrt{2}$ instead of 1.414..., and $\frac{1}{3}$ instead of 0.3333... Explain your reasoning using complete sentences and correct grammar, spelling, and punctuation.

Show ALL of your work!

You have 20 minutes.

Question 1 (6 points). True or false? If true, please explain. If false, construct a counterexample.

(i) Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. If the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent in $\mathbb{R}^n$, then so are $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_k)$ in $\mathbb{R}^m$.

False. Let T be the linear transformation that sends everything to the zero vector.

(The converse is true. If $T(\vec{v}_1), \cdots, T(\vec{v}_k)$ are linearly independent, then

$$c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0} \Rightarrow T(c_1\vec{v}_1 + \cdots + c_k\vec{v}_k) = \vec{0}$$
$$\overset{\text{linearity}}{\Longrightarrow} c_1 T(\vec{v}_1) + \cdots + c_k T(\vec{v}_k) = \vec{0}$$
$$\overset{\text{independency}}{\Longrightarrow} c_1 = \cdots = c_k = 0,$$

which means $\vec{v}_1, \cdots, \vec{v}_k$ are linearly independent.)

(ii) Given an $m \times n$ matrix A with $m > n$, the transformation $\mathbf{x} \mapsto A\mathbf{x}$ cannot be one-to-one.

False. This transformation is one-to-one precisely when the matrix equation $A\vec{x} = \vec{b}$ has at most one solution for each $\vec{b}$ in $\mathbb{R}^m$. This means that the homogeneous equation $A\vec{x} = \vec{0}$ has only the trivial solution. Let

$$A = \left[\begin{array}{ccc} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ \hline & * & \end{array}\right]$$

Question 2 (4 points). Let $T(x_1, x_2) = (3x_1 + x_2, x_1 + h)$ be a transformation from $\mathbb{R}^2$ to itself.

(i) Determine the values of h for which T is onto.

For each $\vec{b} = (b_1, b_2)$ in $\mathbb{R}^2$, we need that

$$\begin{cases} 3x_1 + x_2 = b_1 \\ x_1 + h = b_2 \end{cases}$$

has a solution. Simply take $\begin{cases} x_1 = b_2 - h \\ x_2 = b_1 - 3(b_2 - h) \end{cases}$.

Thus any value of h makes T onto.

(ii) Determine the values of h for which T is a linear transformation. Find its standard matrix in this case.

Given $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{x}' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$, we need

$$T(\vec{x} + \vec{x}') = T(\vec{x}) + T(\vec{x}')$$

Comparing the second components on both sides

$$x_1 + x_1' + h = (x_1 + h) + (x_1' + h)$$

forces $h = 0$. The transformation $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} 3x_1 + x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

is linear, with standard matrix $\begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$.