Question 1 (10 points). Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 7 \end{bmatrix}$$

Is the vector $\mathbf{w}$ in Span$\{\mathbf{v}_1, \mathbf{v}_2\}$?

Solve the equation

$$x_1 \vec{v}_1 + x_2 \vec{v}_2 = \vec{w}$$

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 2 & 1 & 0 \\ -1 & 1 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 3 & -2 \\ 0 & 0 & 8 \end{array}\right]$$ — inconsistent

No solution exists. Thus $\vec{w}$ is not in Span$\{\vec{v}_1, \vec{v}_2\}$.

Question 2 (20 points). It is known that the solution set of a given non-homogeneous linear system $A\mathbf{x} = \mathbf{b}$ of 6 equations in 6 variables has 2 free parameters.

a. How many pivot columns must an echelon form of A have?

Since each parameter corresponds to a column that does not contain a pivot, there are $6-2=4$ pivot columns.

b. Let $\mathbf{a}_1, \ldots, \mathbf{a}_6$ be the column vectors of A. Suppose $\mathbf{c}$ is in Span$\{\mathbf{a}_1, \ldots, \mathbf{a}_6\}$. Is $A\mathbf{x} = \mathbf{c}$ a consistent system? If so, how many free parameters does its solution set have? Explain.

It is consistent, since it has a solution that consists of the weights for $\vec{c}$ as an element in Span$\{\vec{a}_1, \ldots, \vec{a}_6\}$.

There are two parameters, since this system has the same coefficient matrix A, which has two columns without a pivot.

c. Are the column vectors of A linearly dependent or linearly independent? Explain your answer.

They are linearly dependent, since $x_1\vec{a}_1 + \cdots + x_6\vec{a}_6 = \vec{0}$ has nontrivial solutions from above (two "free columns").

d. Do the column vectors of A span $\mathbb{R}^6$? Why or why not?.

No. To span $\mathbb{R}^6$, we need 6 linearly independent vectors, but $\vec{a}_1, \ldots, \vec{a}_6$ are not.

Question 3 (10 points). Solve the following linear system and write the solution set in parametric vector form.

$$\begin{aligned} x_1 + 2x_3 + 3x_4 &= 2 \\ -2x_1 + 2x_2 - 2x_3 - 4x_4 &= -2 \\ -4x_2 - 4x_3 - 4x_4 &= -4 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 2 \\ -2 & 2 & -2 & -4 & -2 \\ 0 & -4 & -4 & -4 & -4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 2 \\ 1 & -1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 2 \\ 0 & -1 & -1 & -1 & -1 \\ 0 & 1 & 1 & 1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 2 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{cases} x_1 = -2s - 3t + 2 \\ x_2 = -s - t + 1 \\ x_3 = s \\ x_4 = t \end{cases} \Rightarrow \vec{x} = s\begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

with s, t parameters

Question 4 (15 points). Let the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ be given by

$$T(x_1, x_2, x_3, x_4) = (x_1 - 2x_2 + 3x_3 + 4x_4, x_2 - x_4, x_1 + x_3).$$

a. Find the standard matrix for T.

$$T(\vec{x}) = \begin{bmatrix} x_1 - 2x_2 + 3x_3 + 4x_4 \\ x_2 - x_4 \\ x_1 + x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{bmatrix}}_{\text{the standard matrix } A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

the standard matrix A

b. Is T onto? Justify your answer.

$$\begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 2 & -2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -2 & -2 \end{bmatrix}$$

Yes. Since A contains a free column, the equation $A\vec{x} = \vec{b}$ has a solution for any $\vec{b}$ in $\mathbb{R}^3$.

c. Is T one-to-one? Justify your answer.

No. Since $A\vec{x} = \vec{0}$ has infinitely many solutions, there are more than one element in $\mathbb{R}^4$ mapping to $\vec{0}$ under T.

Question 5 (10 points). Given an arbitrary 3×4 matrix

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix}$$

with $\mathbf{a}_3 \neq \mathbf{a}_4$, find a matrix B such that

$$AB = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_3 \end{bmatrix}$$

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{B} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_3 \end{bmatrix}$$

Question 6 (10 points). Compute the inverse of the following matrix.

$$\begin{bmatrix} 0 & 1 & 4 \\ 1 & 0 & -3 \\ 2 & 3 & 8 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 4 & 1 & 0 & 0 \\ 1 & 0 & -3 & 0 & 1 & 0 \\ 2 & 3 & 8 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 0 & 1 & 0 \\ 0 & 1 & 4 & 1 & 0 & 0 \\ 2 & 3 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 0 & 1 & 0 \\ 0 & 1 & 4 & 1 & 0 & 0 \\ 0 & 3 & 14 & 0 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 0 & 1 & 0 \\ 0 & 1 & 4 & 1 & 0 & 0 \\ 0 & 0 & 2 & -3 & -2 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 0 & 1 & 0 \\ 0 & 1 & 4 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & -1 & \frac{1}{2} \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{9}{2} & -2 & \frac{3}{2} \\ 0 & 1 & 0 & 7 & 4 & -2 \\ 0 & 0 & 1 & -\frac{3}{2} & -1 & \frac{1}{2} \end{array}\right]$$

The inverse is $\begin{bmatrix} -\frac{9}{2} & -2 & \frac{3}{2} \\ 7 & 4 & -2 \\ -\frac{3}{2} & -1 & \frac{1}{2} \end{bmatrix}$

Question 7 (10 points). Find a basis for Col A, where

$$A = \begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} -3 & 9 & -2 & -7 \\ 0 & 0 & \frac{8}{3} & \frac{10}{3} \\ 0 & 0 & -4 & -5 \end{bmatrix} \sim \begin{bmatrix} -3 & 9 & -2 & -7 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{cccc} \uparrow & & \uparrow & \end{array}$$

pivot columns

A basis for Col A consists of $\begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix}$.

Question 8 (15 points). Decide which of the following statements are true, and justify your answer.

i. If the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, then $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2\} = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

False. Let $\vec{v}_1 = \vec{v}_2 = \vec{0}$ and $\vec{v}_3 \neq \vec{0}$. Then

$$\mathrm{Span}\{\vec{v}_1, \vec{v}_2\} = \{\vec{0}\} \neq \mathrm{Span}\{\vec{v}_3\} = \mathrm{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}.$$

ii. A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ is never one-to-one.

True, because its matrix is 2×3 hence must contain a free column.

iii. Given $A_{5\times 2}$ and $B_{2\times 3}$, each column of the matrix AB can be written as a linear combination of the columns of A.

True. $$AB = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} b_{11}\vec{a}_1 + b_{21}\vec{a}_2 & b_{12}\vec{a}_1 + b_{22}\vec{a}_2 & b_{13}\vec{a}_1 + b_{23}\vec{a}_2 \end{bmatrix}.$$

iv. If $A_{n\times n}$ and $B_{n\times n}$ are given such that $AB = \mathrm{Id}$, then $AB = BA$.

True. Since $AB = \mathrm{Id}$, we know $B = A^{-1}$. Thus $AB = BA = \mathrm{Id}$.

v. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Suppose $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a linearly dependent set in $\mathbb{R}^n$. Then $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_p)\}$ is a linearly dependent set in $\mathbb{R}^m$.

True. Since $\vec{v}_1, \cdots, \vec{v}_p$ are linearly dependent, there exist $c_1, \cdots, c_p$, not all zero, such that

$$c_1\vec{v}_1 + \cdots + c_p\vec{v}_p = \vec{0}.$$

Since T is linear, applying T to the above identity gives

$$c_1T(\vec{v}_1) + \cdots + c_pT(\vec{v}_p) = \vec{0}$$

and thus $T(\vec{v}_1), \cdots, T(\vec{v}_p)$ are linearly dependent.