Question 1 (15 points). Row reduce the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 0 & 1 \\ 3 & -2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

to reduced echelon form. Find the null space and the column space of A by writing down a basis for each.

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 0 & 1 \\ 3 & -2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & -4 & -3 \\ 0 & -8 & -3 \\ 0 & -3 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & \frac{3}{4} \\ 0 & -8 & -3 \\ 0 & -3 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & \frac{3}{4} \\ 0 & 0 & 3 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & \frac{3}{4} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the trivial solution is the only solution to $A\vec{x} = \vec{0}$, the null space $\text{Nul } A = \{\vec{0}\}$. (Its basis is empty.)

In view of the echelon form, the column vectors of A are linearly independent. Thus the column space $\text{Col } A$ has a basis consisting of $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 0 \\ -2 \\ 5 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ 1 \\ 3 \\ 6 \end{bmatrix}$.

Question 2 (10 points). Find all solutions to the following:

$$\begin{aligned} x_1 - x_2 + x_3 &= 3 \\ 2x_2 - x_3 &= 2 \\ 3x_1 - x_2 + 2x_3 &= 11 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & 2 & -1 & 2 \\ 3 & -1 & 2 & 11 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & 2 & -1 & 2 \\ 0 & 2 & -1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 3 \\ 0 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{cases} x_1 = \frac{1}{2}(t+2) - t + 3 = -\frac{1}{2}t + 4 \\ x_2 = \frac{1}{2}(t+2) = \frac{1}{2}t + 1 \\ x_3 = t \end{cases} \quad \text{for all } t$$

Question 3 (10 points). Find the inverse of the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 5 & 2 \\ 1 & 0 & 4 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 2 & 5 & 2 & 0 & 1 & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & -2 & 2 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & -2 & -5 & 2 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & \frac{5}{2} & -1 & -\frac{1}{2} \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -4 & 2 & 1 \\ 0 & 1 & 0 & 3 & -1 & -1 \\ 0 & 0 & 1 & \frac{5}{2} & -1 & -\frac{1}{2} \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -10 & 4 & 3 \\ 0 & 1 & 0 & 3 & -1 & -1 \\ 0 & 0 & 1 & \frac{5}{2} & -1 & -\frac{1}{2} \end{array}\right]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} -10 & 4 & 3 \\ 3 & -1 & -1 \\ \frac{5}{2} & -1 & -\frac{1}{2} \end{bmatrix}$$

Question 4 (10 points). Let

$$\mathbf{a}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$$

Is the vector $\mathbf{b}$ in the span of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$?

If so, there exist x_1, x_2, x_3 such that $\vec{b} = x_1 \vec{a}_1 + x_2 \vec{a}_2 + x_3 \vec{a}_3$.

$$\left[\begin{array}{ccc|c} 2 & 0 & 4 & 5 \\ -1 & 1 & 2 & 0 \\ 1 & -1 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & -2 & 1 \\ -1 & 1 & 2 & 0 \\ 2 & 0 & 4 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & -2 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 2 & 8 & 3 \end{array}\right]$$

In view of the second row in the last matrix, the equation

$$[\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3]\, \vec{x} = \vec{b}$$

has no solution. Thus $\vec{b}$ is not in $\text{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$.

Question 5 (15 point). Find all solutions of the matrix equation $A\mathbf{x} = \mathbf{b}$ in parametric form, where

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 2 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 2 & -1 & 2 & 3 \\ 0 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{cases} x_1 = 1 - t + 1 = 2 - t \\ x_2 = 1 \\ x_3 = t \end{cases} \quad \text{for all } t$$

Question 6 (15 points). True or false? Justify your answers.

(a) A nonhomogeneous equation with at most one free variable has at most one solution.

(b) The equation $A\mathbf{x} = \mathbf{0}$ has at least one nontrivial solution.

(c) If AB is not invertible, then either A or B must not be invertible.

(d) If A is a 3×5 matrix then the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is onto.

(e) If the solution of $A\mathbf{x} = \mathbf{b}$ is unique for every $\mathbf{b}$, then the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is onto.

(a) False. Given a free variable, each of its values corresponds to a solution, and thus there are infinitely many solutions.

(b) False. Not so if A is an identity matrix.

(c) True. Otherwise, both A and B are invertible, and thus $(B^{-1}A^{-1})(AB) = (AB)(B^{-1}A^{-1}) = I$, a contradiction.

(d) False. Not so if A is a zero matrix.

(e) True. In particular, for every $\vec{b}_0$ in the codomain, there exists some $\vec{x}_0$ in the domain such that $T(\vec{x}_0) = A\vec{x}_0 = \vec{b}_0$. Thus T is onto.

Question 7 (15 points). Let the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$T(x_1, x_2, x_3) = (-x_1 + x_2 + x_3, x_2 + 3x_3).$$

(1) Find the standard matrix for T;
(2) Show that T is onto.
(3) Is T one-to-one? Justify your answer.

(1) $$\begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

(2) Given any (b_1, b_2) in $\mathbb{R}^2$,

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & b_1 \\ 0 & 1 & 3 & b_2 \end{array}\right] \Rightarrow \begin{cases} x_1 = -3t + b_2 + t - b_1 = -2t - b_1 + b_2 \\ x_2 = -3t + b_2 \\ x_3 = t \end{cases}$$

and thus $T(-2t - b_1 + b_2, -3t + b_2, t) = (b_1, b_2)$ for all t.
Hence T is onto.

(3) No, since $T(-b_1 + b_2, b_2, 0) = T(-2 - b_1 + b_2, -3 + b_2, 1)$.

Question 8 (10 points). Consider the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Assuming A is invertible, compute $\det A^{-1}$, the determinant of A^{-1}.

Approach 1

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & -\frac{cb}{a}+d & -\frac{c}{a} & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & \frac{c}{bc-ad} & \frac{a}{ad-bc} \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{a} - \frac{b}{a}\left(\frac{c}{bc-ad}\right) & -\frac{b}{ad-bc} \\ 0 & 1 & \frac{c}{bc-ad} & \frac{a}{ad-bc} \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & \frac{-d}{bc-ad} & -\frac{b}{ad-bc} \\ 0 & 1 & \frac{c}{bc-ad} & \frac{a}{ad-bc} \end{array}\right]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} \frac{-d}{bc-ad} & -\frac{b}{ad-bc} \\ \frac{c}{bc-ad} & \frac{a}{ad-bc} \end{bmatrix}$$

$$\Rightarrow \det A^{-1} = \left(\frac{-d}{bc-ad}\right)\left(\frac{a}{ad-bc}\right) - \left(-\frac{b}{ad-bc}\right)\left(\frac{c}{bc-ad}\right) = \frac{1}{ad-bc}$$

Approach 2 Since $\det A = ad - bc$ and $(\det A)(\det A^{-1}) = \det(AA^{-1}) = \det I = 1$, we have $\det A^{-1} = \frac{1}{ad-bc}$.