Question 1 (10 points). Determine the values of h for which the following vectors are linearly independent.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -5 \\ 7 \\ 8 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ h \end{bmatrix}.$$

$$\left[\begin{array}{ccc|c} 1 & -5 & 1 & 0 \\ -1 & 7 & 1 & 0 \\ 3 & 8 & h & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -5 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 23 & h-3 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -5 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & h-26 & 0 \end{array}\right]$$

$h \neq 26$

Question 2 (12 points). Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 3 & 3 \\ -1 & -1 & 1 & 1 \end{bmatrix}$$

a. Reduce A to an Echelon form.

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

b. Compute det(A). Is A invertible? **Hint:**Use Echelon form of A to compute det(A).

$$\det A = 1 \cdot 2 \cdot 2 \cdot 0 = 0$$

A is not invertible.

c. Let T be the linear transformation $T(\mathbf{x}) = A\mathbf{x}$. Is T onto? Justify your answer.

T is not onto because $A\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ has no solution (the bottom row in the echelon form consists of zeros).

d. Find a basis for null space Nul$A$.

$$A\vec{x} = \vec{0}$$

$$\Rightarrow \vec{x} = \begin{bmatrix} -t \\ t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

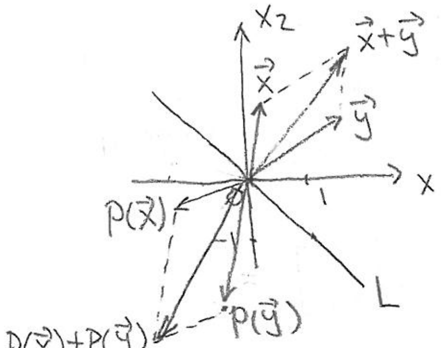
$$\Rightarrow \left\{ \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

Question 3 (10 points). Let $L = \text{Span}\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\}$ be a line in $\mathbb{R}^2$. Let $P : \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection about the line L,

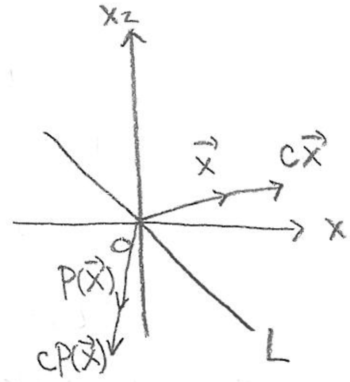
$$P(\mathbf{x}) = \mathbf{x} - 2\mathbf{x}^{\perp}$$

where $\mathbf{x}^{\perp} = \mathbf{x} - \text{proj}_L \mathbf{x}$

a. Show that P is a linear transformation.



$P(\vec{x}+\vec{y}) = P(\vec{x}) + P(\vec{y})$



$P(c\vec{x}) = cP(\vec{x})$

b. Find the standard matrix for P.

$$\left[P\begin{bmatrix} 1 \\ 0 \end{bmatrix} \ \ P\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right] = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

c. Find the eigenvalues and corresponding eigenvectors for the matrix in part b.

$$\det \begin{bmatrix} -\lambda & -1 \\ -1 & -\lambda \end{bmatrix} = \lambda^2 - 1 = 0 \quad \Rightarrow \lambda = \pm 1$$

$\underline{\lambda = 1}$ $\quad \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\underline{\lambda = -1}$ $\quad \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Question 4 (10 points). Suppose for the matrix A we know $\det(A-\lambda I) = \lambda^3(3-\lambda)(4+\lambda)(7+\lambda)(18-\lambda)$

a. Find all possible values of rank A.

Since the characteristic polynomial is of degree 7,
A is 7×7.
Since the algebraic multiplicity of $\lambda = 3, -4, -7, 18$ is 1 each, these eigenvalues have 1-dimensional eigenspaces.
The algebraic multiplicity of $\lambda = 0$ equals 3, and thus the corresponding eigenspace can be of dimension 1, 2 or 3.

examples

dim = 1 $A\vec{x} = \vec{0}$ has 1 free variable, rank $A = 7-1 = 6$ $\begin{bmatrix} 0 & 1 & & & & & \\ & 0 & 1 & & & & \\ & & 0 & & & & \\ & & & 3 & & & \\ & & & & -4 & & \\ & & & & & -7 & \\ & & & & & & 18 \end{bmatrix}$

dim = 2 $A\vec{x} = \vec{0}$ has 2 free variables, rank $A = 7-2 = 5$ $\begin{bmatrix} 0 & 1 & & & & & \\ & 0 & & & & & \\ & & 0 & & & & \\ & & & 3 & & & \\ & & & & -4 & & \\ & & & & & -7 & \\ & & & & & & 18 \end{bmatrix}$

dim = 3 $A\vec{x} = \vec{0}$ has 3 free variables, rank $A = 7-3 = 4$ $\begin{bmatrix} 0 & & & & & & \\ & 0 & & & & & \\ & & 0 & & & & \\ & & & 3 & & & \\ & & & & -4 & & \\ & & & & & -7 & \\ & & & & & & 18 \end{bmatrix}$

b. Answer the same question as in part a. under the further assumption that A is not diagonalizable.

Since A is not diagonalizable, it must be either of the first two cases above
(geometric multiplicity < algebraic multiplicity)

rank A = 6 or 5

Question 5 (10 points). Let

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

a. Find the eigenvalues of A.

$$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} - 1\cdot\begin{vmatrix} 1 & 1 \\ 1 & 2-\lambda \end{vmatrix} + 1\cdot\begin{vmatrix} 1 & 2-\lambda \\ 1 & 1 \end{vmatrix}$$

$$= (2-\lambda)(\lambda^2-4\lambda+3) - (1-\lambda) + (\lambda-1)$$

$$= (2-\lambda)(\lambda-1)(\lambda-3) + 2(\lambda-1)$$

$$= (\lambda-1)(-\lambda^2+5\lambda-4) = -(\lambda-1)^2(\lambda-4) \quad \Rightarrow \lambda = 1 \text{ or } \lambda = 4$$

b. Is A diagonalizable? If so find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

$\underline{\lambda = 1}$ $\quad A-\lambda I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \vec{v} = \begin{bmatrix} -s-t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\underline{\lambda = 4}$ $\quad A-\lambda I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \vec{v} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

A is diagonalizable, $AP = PD$ with $P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

c. Compute P^{-1} in part b.

$$\left[\begin{array}{ccc|ccc} -1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & -1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & -2 & -1 & -1 & 0 \\ 0 & 0 & 3 & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & -2 & -1 & -1 & 0 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right] \Rightarrow P^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Question 6 (10 points). Find an orthonormal basis for Col A, where

$$A = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 2 & 0 \\ 0 & 2 & 3 \\ 1 & 2 & 0 \end{bmatrix}$$

$$\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3$$

$$\vec{v}_1 = \vec{a}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{v}_2 = \vec{a}_2 - \frac{\vec{a}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$$

$$= \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} - \frac{4}{2} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

$$\vec{v}_3 = \vec{a}_3 - \frac{\vec{a}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \frac{\vec{a}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2$$

$$= \begin{bmatrix} 3 \\ 0 \\ 3 \\ 0 \end{bmatrix} - 0 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{12}{8} \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} = \vec{0}$$

Thus an orthogonal basis for Col A is $\{\vec{v}_1, \vec{v}_2\}$

and the corresponding orthonormal basis is $\left\{ \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \right\}$

Question 7 (10 points). Given vectors

$$\mathbf{a} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

find the orthogonal projection of $\mathbf{a}$ onto $\text{Span}\{\mathbf{b}, \mathbf{c}\}$.

Note that $\vec{b}$ and $\vec{c}$ are not orthogonal.

Find an orthogonal basis for $\text{Span}\{\vec{b}, \vec{c}\}$ by taking

$$\vec{v}_1 = \vec{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{v}_2 = \vec{c} - \frac{\vec{c} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Then the orthogonal projection of $\vec{a}$ can be computed as

$$\frac{\vec{a} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 + \frac{\vec{a} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2$$

$$= \frac{4}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{3}{\frac{3}{2}} \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}.$$

Question 8 (10 points). Suppose that a data set consists of points $(-2, 6)$, $(-1, 3)$, $(0, 0)$, $(1, 0)$ and $(2, 1)$ on the xy-plane. Determine the parabola

$$y = ax^2 + bx + c$$

that best models the relation between the x and y coordinates of these sample values. Hint: Compute a least-squares solution for $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 4 & -2 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 6 \\ 3 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Set up the normal equations

$$A^T(\vec{b} - A\vec{x}) = \vec{0}$$

$$A^T A \vec{x} = A^T \vec{b}$$

$$\begin{bmatrix} 4 & 1 & 0 & 1 & 4 \\ -2 & -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & -2 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 4 & 1 & 0 & 1 & 4 \\ -2 & -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 34 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 5 \end{bmatrix} \vec{x} = \begin{bmatrix} 31 \\ -13 \\ 10 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|c} 34 & 0 & 10 & 31 \\ 0 & 10 & 0 & -13 \\ 10 & 0 & 5 & 10 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 0 & 1 & 2 \\ 0 & 10 & 0 & -13 \\ 0 & 0 & -7 & -3 \end{array}\right] \Rightarrow \vec{x} = \begin{bmatrix} \frac{1}{2}(2 - \frac{3}{7}) \\ -\frac{13}{10} \\ \frac{3}{7} \end{bmatrix} = \begin{bmatrix} \frac{11}{14} \\ -\frac{13}{10} \\ \frac{3}{7} \end{bmatrix}.$$

Thus the parabola is $y = \frac{11}{14}x^2 - \frac{13}{10}x + \frac{3}{7}$.

Question 9 (18 points). True or false? Justify your answer

a. A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^4$ is never onto.

True, $T\begin{bmatrix}1\\0\end{bmatrix}$ and $T\begin{bmatrix}0\\1\end{bmatrix}$ can span a subspace of $\mathbb{R}^4$ of dimension at most 2.

b. If $\mathbf{v}$ and $\mathbf{w}$ are two eigenvectors for the matrix A then $2\mathbf{v} + 3\mathbf{w}$ must also be an eigenvector for A.

False, if $\vec{v}$ and $\vec{w}$ belong to different eigenvalues.

Even better, let $A = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$, $\vec{v} = \begin{bmatrix}1\\0\end{bmatrix}$, $\vec{w} = \begin{bmatrix}0\\1\end{bmatrix}$.

c. Every real 2×2 matrix with complex eigenvalues with non-zero imaginary part is similar to a matrix of rotation around the origin by some angle θ.

False, in general it is similar to the product of such a matrix and a diagonal matrix $\begin{bmatrix}c & 0\\0 & c\end{bmatrix}$ (dilation).

d. If A^3 is not invertible, neither is A.

True. Suppose A is invertible, with inverse A^{-1}. Then $(A^{-1})^3$ would be the inverse of A^3.

e. A 4×4 real matrix always has at least one real eigenvalue.

False. Let $A = \left[\begin{array}{cc|cc} 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ \hline 0 & 0 & 0 & -2\\ 0 & 0 & 2 & 0\end{array}\right]$.

f. If two $n \times n$ matrices A and B have the same characteristic polynomials then they are similar.

False. Let $A = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$ and $B = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$.