Question 1. Solve the matrix equation $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -3 & -1 & 2 \\ 0 & 5 & 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ -3 & -1 & 2 & 1 \\ 0 & 5 & 3 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 5 & 5 & 1 \\ 0 & 5 & 3 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 5 & 5 & 1 \\ 0 & 0 & -2 & -2 \end{array}\right]$$

$$\Rightarrow \begin{cases} x_1 = -2x_2 - x_3 = \frac{8}{5} - 1 = \frac{3}{5} \\ x_2 = \frac{1}{5}(1 - 5x_3) = -\frac{4}{5} \\ x_3 = 1 \end{cases}$$

Question 2. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation that reflects each vector through the plane $x_2 = 0$. Find the standard matrix of T.

$$T(x_1, x_2, x_3) = (x_1, -x_2, x_3)$$

The standard matrix is given by

$$\begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Question 3. Find a basis for Nul A, where

$$A = \begin{bmatrix} -3 & 9 & -2 & -7 \\ 2 & -6 & 4 & 8 \\ 3 & -9 & -2 & 2 \end{bmatrix}$$

What is the dimension of Nul A?

$$A \sim \begin{bmatrix} -3 & 9 & -2 & -7 \\ 1 & -3 & 2 & 4 \\ 3 & -9 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 4 \\ -3 & 9 & -2 & -7 \\ 0 & 0 & -4 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 2 & 4 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3s - 2(-\frac{5}{4}t) - 4t \\ s \\ -\frac{5}{4}t \\ t \end{bmatrix} = s\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -\frac{3}{2} \\ 0 \\ -\frac{5}{4} \\ 1 \end{bmatrix}$$

A basis for Nul A consists of $\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3/2 \\ 0 \\ -5/4 \\ 1 \end{bmatrix}$, with

dim Nul A = 2.

Question 4. True or false? Justify your answer.

Given any $m \times n$ matrix A, the number of linearly independent columns equals the number of linearly independent rows.

True.

Turn A into a reduced echelon form by row operations, which preserve these two numbers. Both numbers then equal the number of pivots.

Question 5. Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}.$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ \frac{7}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

Question 6. Compute the determinant of the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 & 6 \\ 1 & -7 & -5 & 0 \\ 3 & 8 & 6 & 0 \\ 0 & 7 & 5 & 4 \end{bmatrix}$$

Is A invertible?

$$\det A = 2 \begin{vmatrix} -7 & -5 & 0 \\ 8 & 6 & 0 \\ 7 & 5 & 4 \end{vmatrix} - 6 \begin{vmatrix} 1 & -7 & -5 \\ 3 & 8 & 6 \\ 0 & 7 & 5 \end{vmatrix}$$

$$= 2 \cdot 4 \begin{vmatrix} -7 & -5 \\ 8 & 6 \end{vmatrix} - 6\left(1 \cdot \begin{vmatrix} 8 & 6 \\ 7 & 5 \end{vmatrix} - 3 \begin{vmatrix} -7 & -5 \\ 7 & 5 \end{vmatrix}\right)$$

$$= 8(-42+40) - 6(40-42)$$

$$= -16 + 12 = -4$$

Since $\det A \neq 0$, A is invertible.

Question 7. Consider the following matrix.

$$A = \begin{bmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{bmatrix}$$

(i) Compute the characteristic polynomial of A.

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 0 & -2 \\ 2 & 5-\lambda & 4 \\ 0 & 0 & 5-\lambda \end{vmatrix}$$

$$= (4-\lambda)\begin{vmatrix} 5-\lambda & 4 \\ 0 & 5-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & 5-\lambda \\ 0 & 0 \end{vmatrix}$$

$$= (4-\lambda)(5-\lambda)^2$$

(ii) Find all the eigenvalues and eigenvectors of A.

$\det(A - \lambda I) = 0 \Rightarrow \lambda = 4$ or $\lambda = 5$

Thus the eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = 5$.

$\underline{\lambda_1 = 4}$

$$A - \lambda_1 I = \begin{bmatrix} 0 & 0 & -2 \\ 2 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{t}{2} \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}$$

$\underline{\lambda_2 = 5}$

$$A - \lambda_2 I = \begin{bmatrix} -1 & 0 & -2 \\ 2 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

(iii) Write $A = PDP^{-1}$ with P invertible and D diagonal.

$$A = \begin{bmatrix} -1/2 & 0 & -2 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} -1/2 & 0 & -2 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

Question 8. Let $\mathbf{v}$ be any nonzero vector in $\mathbb{R}^n$ with $n \geq 2$. Find all the eigenvalues of the $n \times n$ matrix $\mathbf{v}\,\mathbf{v}^T$.

By definition, if λ is an eigenvalue, then $\vec{v}\,\vec{v}^T\,\vec{w} = \lambda\,\vec{w}$ for some nonzero $\vec{w}$. Since

$$\lambda\,\vec{w} = \vec{v}\,\vec{v}^T\,\vec{w} = (\vec{v}\cdot\vec{w})\,\vec{v}$$

we have $\vec{w} = \dfrac{\vec{v}\cdot\vec{w}}{\lambda}\,\vec{v}$ if $\lambda \neq 0$. Write $c = \dfrac{\vec{v}\cdot\vec{w}}{\lambda} \neq 0$.

We then have

$$\lambda\, c\,\vec{v} = (\vec{v}\cdot c\vec{v})\,\vec{v}$$

and thus $\lambda = \vec{v}\cdot\vec{v}$ is an eigenvalue. On the other hand, if $\lambda = 0$, then any nonzero $\vec{w}$ in $\operatorname{Span}\{\vec{v}\}^{\perp}$ serves as an eigenvector. Such a $\vec{w}$ exists because $n \geq 2$. Hence the eigenvalues are $\vec{v}\cdot\vec{v}$ and 0.

Question 9. Find an orthonormal basis for Col A, where

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Denote the column vectors by $\vec{a}_1, \vec{a}_2, \vec{a}_3$. Then

$$\vec{v}_1 = \vec{a}_1$$

$$\vec{v}_2 = \vec{a}_3$$

$$\vec{v}_3 = \vec{a}_2 - \frac{\vec{a}_2\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1 - \frac{\vec{a}_2\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\vec{v}_2$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

form an orthogonal basis for Col A. An orthonormal basis then consists of

$$\vec{u}_1 = \frac{1}{\sqrt{2}}\vec{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \quad \vec{u}_2 = \frac{1}{\sqrt{2}}\vec{v}_2 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \quad \vec{u}_3 = \vec{v}_3 = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

Question 10. Suppose that a data set consists of points $(-6,-1)$, $(-2,2)$, $(1,1)$ and $(7,6)$ on the xy-plane. Find an equation for the line that best models the relation between the x and y coordinates of these sample values. Hint: Compute a least-squares solution for $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} -6 & 1 \\ -2 & 1 \\ 1 & 1 \\ 7 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 6 \end{bmatrix}.$$

Suppose that the line is given by $y = kx + t$. Then $\begin{bmatrix} k \\ t \end{bmatrix}$ is a least-squares solution for $A\vec{x} = \vec{b}$. Solve the normal equations $A^T(\vec{b} - A\vec{x}) = \vec{0}$

$$A^T A \vec{x} = A^T \vec{b}$$

$$\begin{bmatrix} -6 & -2 & 1 & 7 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -6 & 1 \\ -2 & 1 \\ 1 & 1 \\ 7 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} -6 & -2 & 1 & 7 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 90 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} k \\ t \end{bmatrix} = \begin{bmatrix} 45 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} k \\ t \end{bmatrix} = \begin{bmatrix} 1/2 \\ 2 \end{bmatrix}$$

Thus $y = \frac{1}{2}x + 2$.