Change of bases and similarity of matrices

Recall Given a basis $\mathcal{U} = \{\vec{u}_1, \cdots, \vec{u}_n\}$ of $\mathbb{R}^n$, for any vector $\vec{x}$ in $\mathbb{R}^n$, the coordinates of $\vec{x}$ with respect to $\mathcal{U}$ are the weights that appear in the unique expression of $\vec{x}$ as a linear combination of $\vec{u}_1, \cdots, \vec{u}_n$:

$$\vec{x} = x_1 \vec{u}_1 + \cdots + x_n \vec{u}_n$$

They give a column vector denoted by $[\vec{x}]_{\mathcal{U}}$

$$[\vec{x}]_{\mathcal{U}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Given a linear transformation $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$, the matrix A that represents T with respect to $\mathcal{U}$ is given by

$$A = \begin{bmatrix} [T(\vec{u}_1)]_{\mathcal{U}} & \cdots & [T(\vec{u}_n)]_{\mathcal{U}} \end{bmatrix}$$

so that

$$\begin{aligned} [T(\vec{x})]_{\mathcal{U}} &= \Big[T\big([\vec{u}_1 \cdots \vec{u}_n][\vec{x}]_{\mathcal{U}} \big) \Big]_{\mathcal{U}} \\ &= \Big[[T(\vec{u}_1) \cdots T(\vec{u}_n)][\vec{x}]_{\mathcal{U}} \Big]_{\mathcal{U}} \\ &= \Big[[T(\vec{u}_1)]_{\mathcal{U}} \cdots [T(\vec{u}_n)]_{\mathcal{U}} \Big] [\vec{x}]_{\mathcal{U}} \\ &= A [\vec{x}]_{\mathcal{U}} \end{aligned}$$

(This generalizes the standard matrix of T, where the underlying basis is the standard basis $\{\vec{e}_1, \cdots, \vec{e}_n\}$.)

Now Let $\mathcal{V} = \{\vec{v}_1, \dots, \vec{v}_n\}$ be another basis, with

$$[\vec{v}_1 \cdots \vec{v}_n] = [\vec{u}_1 \cdots \vec{u}_n]\, C$$

for some invertible C. Let B be the matrix that represents T with respect to $\mathcal{V}$.

Claim The matrices A and B are similar; $B = C^{-1} A C$

Proof Since $\vec{x} = [\vec{u}_1 \cdots \vec{u}_n]\, [\vec{x}]_{\mathcal{U}}$

$$= [\vec{v}_1 \cdots \vec{v}_n]\, C^{-1}\, [\vec{x}]_{\mathcal{U}}$$

we have $[\vec{x}]_{\mathcal{V}} = C^{-1} [\vec{x}]_{\mathcal{U}}$ (by uniqueness of the coordinates with respect to $\mathcal{V}$). Thus

$$\begin{aligned}
B &= \Big[\, [T(\vec{v}_1)]_{\mathcal{V}} \cdots [T(\vec{v}_n)]_{\mathcal{V}} \,\Big] \\
&= \Big[\, C^{-1}[T(\vec{v}_1)]_{\mathcal{U}} \cdots C^{-1}[T(\vec{v}_n)]_{\mathcal{U}} \,\Big] \\
&= C^{-1} \Big[\, [T(\vec{v}_1)]_{\mathcal{U}} \cdots [T(\vec{v}_n)]_{\mathcal{U}} \,\Big] \\
&= C^{-1} \Big[\, [T(\vec{u}_1)]_{\mathcal{U}} \cdots [T(\vec{u}_n)]_{\mathcal{U}} \,\Big]\, C \\
&= C^{-1} A C
\end{aligned}$$