Claim Eigenvectors of distinct eigenvalues are linearly independent.

Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be distinct eigenvalues of the matrix A (or more intrinsically, of the linear transformation T that A represents with respect to a chosen basis).

Let $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ be their respective eigenvectors.

Want to show that $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ are linearly independent, that is, by definition,

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0} \quad \text{forces each } c_i \text{ must be zero.}$$

Suppose not. Reorder and choose

$$(*) \qquad c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0}$$

for some $1 \leq k \leq n$ such that each c_i is not zero (possible if $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ were linearly dependent).

Now multiply the matrix A (or apply the transformation T) to (*) and get

$$c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 + \cdots + c_k\lambda_k\vec{v}_k = \vec{0}.$$

Iterate this process and get

$$c_1\lambda_1^2\vec{v}_1 + c_2\lambda_2^2\vec{v}_2 + \cdots + c_k\lambda_k^2\vec{v}_k = \vec{0}$$

$$\cdots$$

$$c_1\lambda_1^{k-1}\vec{v}_1 + c_2\lambda_2^{k-1}\vec{v}_2 + \cdots + c_k\lambda_k^{k-1}\vec{v}_k = \vec{0}$$

Together with (*), each of these identities gives a column

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \end{bmatrix}\begin{bmatrix} c_1\lambda_1^i \\ c_2\lambda_2^i \\ \vdots \\ c_k\lambda_k^i \end{bmatrix} = \vec{0} \qquad 0 \le i \le k-1$$

Put together, we have

$$(**) \qquad \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \end{bmatrix}\begin{bmatrix} c_1 & c_1\lambda_1 & c_1\lambda_1^2 & \cdots & c_1\lambda_1^{k-1} \\ c_2 & c_2\lambda_2 & c_2\lambda_2^2 & \cdots & c_2\lambda_2^{k-1} \\ \vdots & \vdots & \vdots & & \vdots \\ c_k & c_k\lambda_k & c_k\lambda_k^2 & \cdots & c_k\lambda_k^{k-1} \end{bmatrix} = \begin{bmatrix} O \end{bmatrix}_{k\times k}$$

The contradiction comes from the <u>invertibility</u> of the second matrix above. Indeed,

$$\det\begin{bmatrix} c_1 & c_1\lambda_1 & c_1\lambda_1^2 & \cdots & c_1\lambda_1^{k-1} \\ c_2 & c_2\lambda_2 & c_2\lambda_2^2 & \cdots & c_2\lambda_2^{k-1} \\ \vdots & \vdots & \vdots & & \vdots \\ c_k & c_k\lambda_k & c_k\lambda_k^2 & \cdots & c_k\lambda_k^{k-1} \end{bmatrix}$$

$$= c_1c_2\cdots c_k \boxed{\det\begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{k-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{k-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \lambda_k & \lambda_k^2 & \cdots & \lambda_k^{k-1} \end{bmatrix}} \quad \text{the Vandermonde determinant}$$

$$= \prod_{i=1}^{k} c_i \prod_{i>j} (\lambda_i - \lambda_j) \neq 0$$

This matrix being invertible, together with the equation $(**)$, implies

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \end{bmatrix} = 0$$

but $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k$ are nonzero eigenvectors.